Hodge Theory, Algebraic Cycles of Hyper-Kähler Manifolds

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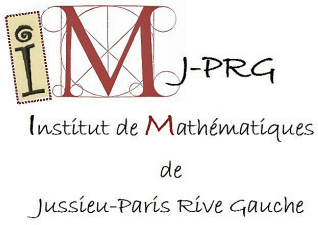

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présentée par

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Hodge Theory, Algebraic Cycles of Hyper-Kähler Manifolds

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dirigée par Claire VOISIN

Soutenance le 14 juin 2024 devant le jury composé de:

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Résumé

Cette thèse est consacrée à l'étude des cycles algébriques dans les variétés hyper-Kähleriennes projectives et les variétés de Calabi-Yau strictes. Elle contribue à la compréhension des conjectures de Beauville et de Voisin sur les anneaux de Chow des variétés hyper-kählériennes projectives et des variétés de Calabi-Yau strictes. Elle étudie également certains invariants birationnels des variétés hyper-kählériennes projectives.

La première partie de la thèse, parue dans Mathematische Zeitschrift [Bai23] et présentée dans le chapitre 2, étudie si les sous-variétés lagrangiennes dans une variété hyper-kählérienne partageant la même classe cohomologique ont également la même classe de Chow. Nous étudions la notion de familles lagrangiennes et ses applications aux applications d'Abel-Jacobi associées. Nous adoptons une approche infinitésimale pour donner un critère de trivialité de l'application d'Abel-Jacobi d'une famille lagrangienne, et utilisons ce critère pour donner une réponse négative à la question précédente, ajoutant aux subtilités d'une conjecture de Voisin. Nous explorons également comment la maximalité de la variation des structures de Hodge sur la cohomologie de degré 1 de la famille lagrangienne implique la trivialité de l'application d'Abel-Jacobi.

La deuxième partie de la thèse, parue dans International Mathematics Research Notices [Bai24] et présentée dans le chapitre 3, étudie le degré d'irrationalité, la gonalité fibrante et le genre fibrant des variétés hyper-kählériennes projectives. Nous commençons par donner une légère amélioration d'un résultat de Voisin sur la borne inférieure du degré d'irrationalité des variétés hyper-kählériennes générales de Mumford-Tate. Nous étudions ensuite la relation entre les trois invariants birationnels susmentionnés pour les surfaces K3 projectives de nombre de Picard 1, rajoutant la compréhension sur une conjecture de Bastianelli, De Poi, Ein, Lazarsfeld, Ullery sur le comportement asymptotique du degré d'irrationalité des surfaces K3 projectives très générales.

La troisième partie de la thèse, parue dans ArXiv [Bai24II], présentée dans le chapitre 4, étudie les applications de Voisin de dimension supérieure sur les variétés de Calabi-Yau strictes. Voisin a construit des applications auto-rationnelles de variétés de Calabi-Yau obtenues comme des variétés de r-plans dans des hypersurfaces cubiques de dimension adéquate. Cette application a été largement étudiée dans le cas $r = 1$, qui est le cas de Beauville-Donagi. Dans les cas de dimensions supérieures, nous étudions d'abord l'action de l'application de Voisin sur les formes holomorphes. Nous demontrons ensuite la conjecture de Bloch généralisée pour l'action des applications de Voisin sur les groupes de Chow dans le cas de $r = 2$. Enfin, via l'étude de l'application de Voisin, nous apportons des éléments de preuve à une conjecture de Voisin sur l'existence d'un 0-cycle spécial sur les variétés de Calabi-Yau strictes.

Mots-clés : Variétés hyper-kählériennes, Variétés de Calabi-Yau strictes, Conjectures de Voisin, Familles lagrangiennes, Degré d'irrationalité, Applications de Voisin

Abstract

This thesis is devoted to the study of algebraic cycles in projective hyper-Kähler manifolds and strict Calabi-Yau manifolds. It contributes to the understanding of Beauville's and Voisin's conjectures on the Chow rings of projective hyper-Kähler manifolds and strict Calabi-Yau manifolds. It also studies some birational invariants of projective hyper-Kähler manifolds.

The first part of the thesis, appeared in Mathematische Zeitschrift [Bai23] and presented in Chapter 2, studies whether the Lagrangian subvarieties in a hyper-Kähler manifold sharing the same cohomological class have the same Chow class as well. We study the notion of Lagrangian families and its associated Abel-Jacobi maps. We take an infinitesimal approach to give a criterion for the triviality of the Abel-Jacobi map of a Lagrangian family, and use this criterion to give a negative answer to the above question, adding to the subtleties of a conjecture of Voisin. We also explore how the maximality of the variation of the Hodge structures on the degree 1 cohomology the Lagrangian family implies the triviality of the Abel-Jacobi map.

The second part of the thesis, appeared in International Mathematics Research Notices [Bai24] and presented in Chapter 3, studies the degree of irrationality, the fibering gonality and the fibering genus of projective hyper-Kähler manifolds, with emphasis on the K3 surfaces case, en mettant l'accent sur le cas des surfaces K3. We first give a slight improvement of a result of Voisin on the lower bound of the degree of irrationality of Mumford-Tate general hyper-Kähler manifolds. We then study the relation of the above three birational invariants for projective K3 surfaces of Picard number 1, adding the understandinf of a conjecture of Bastianelli, De Poi, Ein, Lazarsfeld, Ullery on the asymptotic behavior of the degree of irrationality of very general projective K3 surfaces.

The third part of the thesis, appeared in ArXiv [Bai24II], presented in Chapter 4, studies the higher dimensional Voisin maps on strict Calabi-Yau manifolds. Voisin constructed self-rational maps of Calabi-Yau manifolds obtained as varieties of r-planes in cubic hypersurfaces of adequate dimension. This map has been thoroughly studied in the case $r = 1$, which is the Beauville-Donagi case. For higher dimensional cases, we first study the action of the Voisin map on the holomorphic forms. We then prove the generalized Bloch conjecture for the action of the Voisin maps on Chow groups for the case of $r = 2$. Finally, via the study of the Voisin map, we provide evidence for a conjecture of Voisin on the existence of a special 0-cycle on strict Calabi-Yau manifolds.

Keywords: Hyper-Kähler manifolds, Strict Calabi-Yau manifolds, Voisin conjectures, Lagrangian families, Degree of irrationality, Voisin maps

Contents

Chapter 1

Introduction

1.1 Chow groups

We closely follow the presentations in [Ful84], [Voi03, Chapter 17], and [Voi14, Chapter 2]. Let X be an algebraic variety of dimension n, defined over a field which will be in this thesis the field of complex numbers. An algebraic cycle $Z \subset X$ of dimension k is defined as a finite formal sum $Z = \sum_i n_i Z_i$ with $n_i \in \mathbb{Z}$, and Z_i is a closed irreducible subvariety of X of dimension k. The cycle group $\mathscr{Z}_k(X)$ is the abelian group of all algebraic cycles of dimension k in X. If $\phi : X \to Y$ is a proper morphism, there is an induced map $\phi_* : \mathscr{Z}_k(X) \to \mathscr{Z}_k(Y)$, called the push-forward map, defined as follows: for an irreducible subvariety $Z \subset X$ of dimension k with $\phi|_Z : Z \to Z'$ being a generically finite map of degree d, we define $\phi_* Z = dZ'$. In other cases, $\phi_* Z = 0$. Finally, we extend the definition of ϕ_* linearly to the entire group $\mathscr{Z}_k(X)$.

Definition 1.1.1. Two algebraic cycles $Z_1, Z_2 \subset X$ of dimension k are called rationally equivalent if there exist irreducible subvarieties $W_1, \ldots, W_l$ of dimension $k+1$ in X, nonzero rational functions f_i on $\tilde{W}_i$ where $\tau_i : \tilde{W}_i \to W_i$ is the normalization of W_i, such that

$$Z_1 - Z_2 = \sum_{i=1}^{l} \tau_{i*}(\mathrm{div}(f_i)).$$

Notation 1.1.2. Throughout this thesis, we use the following notation. Let X be an algebraic variety of dimension n.

- The Chow group $CH_k(X) = \mathscr{Z}_k(X)/\equiv_{rat}$ is the quotient group of $\mathscr{Z}_k(X)$ modulo rational equivalence $\equiv_{rat}$.
- $\mathscr{Z}^k(X) = \mathscr{Z}_{n-k}(X)$.
- $CH^k(X) = CH_{n-k}(X)$.
- $CH_k(X)_{\mathbb{Q}} = CH_k(X) \otimes_{\mathbb{Z}} \mathbb{Q}$.
- $CH^k(X)_{\mathbb{Q}} = CH^k(X) \otimes_{\mathbb{Z}} \mathbb{Q}$.

In [Ful84, Chapter 6], an intersection product is constructed for smooth algebraic varieties.

Theorem 1.1.3 (Fulton)**.** *Let X be a smooth quasi-projective variety. There exists a unique product structure on $CH^*(X)$ satisfying the following condition: if two subvarieties A,B are transverse to each other, then $[A].[B] = [A \cap B] \in CH^*(X)$. This product structure makes $CH^*(X)$ a graded ring.*

1.1.1 Push-forward and pull-back maps

Let $p : X \to Y$ be a flat morphism of relative dimension l. Then, there is a naturally defined map $\phi^* : \mathscr{Z}_k(Y) \to \mathscr{Z}_{k+l}(X)$, called the pull-back map, defined by taking the preimage. A fundamental fact is the following (See Lemma 2.3 and 2.5 in [Voi14]).

Lemma 1.1.4. *The push-forward map and the pull-back map, defined at the level of cycle groups, preserve rational equivalence.*

Therefore, the push-forward map of a proper morphism and the pull-back map of a flat morphism can be defined at the level of Chow groups. Let $p : X \to Y$ be a morphism between smooth varieties (in fact, we only need Y to be smooth). We can also define the pull-back map $p^* : CH^k(Y) \to CH^k(X)$ without flatness conditions (see [Voi03, 17.2]). We have the following functoriality result.

Proposition 1.1.5. *Let $p : X \to Y$ be a morphism between smooth varieties.*
(a) (Projection formula) For $Z \in CH(Y)$ and $Z' \in CH(X)$, we have

$$p_*(p^*Z.Z') = Z.p_*(Z') \in CH(X).$$

(b) For $Z,Z' \in CH(Y)$, we have

$$p^*(Z.Z') = p^*Z.p^*Z' \in CH(X).$$

Proof. See [Ful84, 8.1]. □

A correpondence between X and Y is an element Z in $CH(X \times Y)$. Let $p_X : X \times Y \to X$ and $p_Y : X \times Y \to Y$ be the two projection maps. A correspondence Z between smooth varieties X and Y induces two natural maps $Z_* : CH(X) \to CH(Y)$ and $Z^* : CH(Y) \to CH(X)$ in the following way. For any $z \in CH(X)$,

$$Z_*x := p_{Y,*}(p_X^*x.Z) \in CH(Y),$$

and for any $w \in CH(Y)$,

$$Z^*w := p_{X,*}(p_Y^*w.Z) \in CH(X).$$

1.1.2 Localization exact sequence

Let X be a quasi-projective variety, and let $j : Y \hookrightarrow X$ be a closed algebraic subset. Let $i : U := X - Y \hookrightarrow X$ be the open embedding of the complement of Y.

Proposition 1.1.6. *We have the following exact sequence:*

$$CH_k(Y) \xrightarrow{j_*} CH_k(X) \xrightarrow{i^*} CH_k(U) \to 0.$$

Proof. See [Voi03, Lemma 17.12]. □

1.1.3 Cycle class map

Let X be a smooth complex algebraic variety. To each irreducible subvariety $Z \subset X$ of codimension k, we can associate its cohomological class in $H^{2k}(X,\mathbb{Z})$. Extending linearly, we get the cycle class map:

$$cl : \mathscr{Z}^k(X) \to H^{2k}(X,\mathbb{Z}).$$

Lemma 1.1.7. *Let $Z_1, Z_2 \in \mathscr{Z}^k(X)$. If Z_1 and Z_2 are rationally equivalent, then $cl(Z_1) = cl(Z_2)$ in $H^{2k}(X,\mathbb{Z})$.*

Thus, the cycle class map descends to the Chow group level:

$$cl : CH^k(X) \to H^{2k}(X,\mathbb{Z}),$$

and we still call it the cycle class map.

We have the following compatibility results of the cycle class map (see [Voi03, 17.2.4]).

Proposition 1.1.8. *The cycle class map $cl : CH^*(X) \to H^{2k}(X,\mathbb{Z})$ is a ring homomorphism.*

Proposition 1.1.9. *Let $p : X \to Y$ be a morphism between smooth varieties.*
(a) For any $Z \in CH^k(Y)$, we have

$$p^* cl(Z) = cl(p^* Z) \in H^{2k}(X,\mathbb{Z}).$$

(b) If p is proper, then for any $Z \in CH_k(X)$, we have

$$p_* cl(Z) = cl(p_* Z) \in H^{2\dim Y - 2k}(Y,\mathbb{Z}).$$

Notation 1.1.10. The kernel of the cycle class map $cl : CH^k(X) \to H^{2k}(X,\mathbb{Z})$ is denoted by $CH^k(X)_{hom}$. The subscript "hom" signifies "homologous to 0". Similarly, $CH^k(X)_{\mathbb{Q},hom}$ denotes the kernel of $cl : CH^k(X)_{\mathbb{Q}} \to H^{2k}(X,\mathbb{Q})$.

1.1.4 Constant cycle subvarieties

The notion of constant cycle subvarieties is introduced and developed in [Huy14, Voi16], especially in the context of algebraic hyper-Kähler manifold.

Definition 1.1.11 ([Huy14, Voi16])**.** Let X be an algebraic variety. A subvariety $i : Z \hookrightarrow X$ is called a constant cycle subvariety if every two points $z_1, z_2 \in Z$ are rationally equivalent in X. In other words, the image of the push-forward map

$$i_* : CH_0(Z) \to CH_0(X)$$

is $\mathbb{Z}$.

Remark 1.1.12. The study of rational curves in an algebraic variety is a powerful method to understand the algebraic variety itself ([Kol96]). Constant cycle subvarieties are a big generalization of rationally connected subvarieties. The existence of such subvarieties often gives many interesting geometric implications (e.g., [Voi16, Lin16, Baz17]). On the other hand, constant cycle subvarieties share similar properties to rationally connected subvarieties. For example, it is classical that there are no rational curves in abelian varieties. One can also prove that there are no constant cycle subvarieties of positive dimension in abelian varieties either.

Constant cycle subvarieties will appear in Chapter 4 where we prove the following theorem (see Theorem 4.3.26).

Theorem 1.1.13. *Let $Y \subset \mathbb{P}^9$ be a general cubic eightfold. Let $X = F_2(Y)$ be its Fano variety of planes. Let $F \subset X$ be the closure of the set of the points $x \in X$ parametrizing the planes $P_x \subset Y$ such that there exists a unique linear subspace H of dimension 3 such that $H \cap Y = 3P_x$. Then $F \subset X$ is a constant cycle subvariety of X of codimension 3.*

This theorem serves as a crucial ingredient for our understanding of the Voisin maps in Chapter 4.

1.1.5 Bloch-Beilinson filtration

It has been conjectured by Bloch and Beilinson that there exists a decreasing filtration $F^iCH_k(X)_{\mathbb{Q}}$ on the Chow groups with rational coefficients of any smooth complex projective variety X, satisfying several axioms. The precise statements are as follows. We follow closely the presentation in [Voi04II] and [Voi14, Conjecture 2.19] for the statements of the Bloch-Beilinson conjectures.

Conjecture 1.1.14 (Bloch-Beilinson Conjecture)**.** *For any smooth projective variety X, there exists a decreasing filtration $F^\bullet$ on $CH^i(X)_{\mathbb{Q}}$, with the following properties:*
(i) (Non-Triviality) $F^0CH^i(X)_{\mathbb{Q}} = CH^i(X)_{\mathbb{Q}}$ and $F^1CH^i(X)_{\mathbb{Q}} = CH^i(X)_{\mathbb{Q},hom}$.
(ii) (Functoriality) If $Z \in CH^k(X \times Y)_{\mathbb{Q}}$, then $Z_*(F^iCH^l(X)_{\mathbb{Q}}) \subset F^iCH^{l+k-n}(X)_{\mathbb{Q}}$, where $n = \dim X$.
(iii) (Graded) The induced map $Z_* : Gr_F^iCH^l(X)_{\mathbb{Q}} \to Gr_F^iCH^{l+k-n}(Y)_{\mathbb{Q}}$ vanishes for any i if $[Z] = 0$ in $H^{2k}(X \times Y, \mathbb{Q})$.
(iv) (Finiteness) One has $F^{k+1}CH^k(X)_{\mathbb{Q}} = 0$ for any X and k.

There is also a strengthened version of the Bloch-Beilinson conjecture where (iii) above is replace by

(iii)' (Graded) For a fixed i, the induced map $Z_* : Gr_F^iCH^l(X)_{\mathbb{Q}} \to Gr_F^iCH^{l+k-n}(Y)_{\mathbb{Q}}$ vanishes if the following induced map

$$[Z]^* : H^{2m-2k-2l+2n+i}(Y, \mathbb{Q}) \to H^{2n-2l+i}(X, \mathbb{Q})$$

vanishes, where $m = \dim Y$ and $n = \dim X$.

1.2 Interaction between Hodge structures and Chow groups

1.2.1 Hodge Structures and Coniveau

Definition 1.2.1. A weight k Hodge structure (respectively, a rational Hodge structure) on H consists of a free abelian group $H_{\mathbb{Z}}$ (respectively, a $\mathbb{Q}$-vector space $H_{\mathbb{Q}}$) and a decomposition

$$H_{\mathbb{C}} := H \otimes \mathbb{C} = \bigoplus_{p+q=k} H^{p,q},$$

satisfying the Hodge symmetry condition

$$\overline{H^{p,q}} = H^{q,p}.$$

Hodge structures naturally arise from the Betti cohomology groups of compact Kähler manifolds [Voi03, 7.1].

Theorem 1.2.2 (Hodge Decomposition Theorem [Voi03])**.** *Let X be a compact Kähler manifold. Then the k-th Betti cohomology group $H^k(X,\mathbb{Z})$ of X possesses a weight k Hodge structure. Specifically, there is a decomposition*

$$H^k(X,\mathbb{C}) = \bigoplus_{p+q=k} H^{p,q}(X),$$

such that $\overline{H^{p,q}(X)} = H^{q,p}(X)$. Moreover, there is a canonical isomorphism $H^q(X,\Omega_X^p) \cong H^{p,q}(X)$.

Definition 1.2.3. A (rational) Hodge class in a (rational) Hodge structure H of degree $2k$ is an element in $H_{\mathbb{Z}} \cap H^{k,k}$ (respectively, $H_{\mathbb{Q}} \cap H^{k,k}$).

Definition 1.2.4. The (Hodge) coniveau $c \leq k/2$ of a weight k Hodge structure $(H_{\mathbb{Z}}, H^{p,q})$ is the smallest integer p for which $H^{p,q} \neq 0$.

A multitude of conjectures relate algebraic cycles and Hodge structures.

Conjecture 1.2.5 (Hodge Conjecture)**.** *Let X be a complex smooth projective variety. Then any rational Hodge class in $H^{2k}(X,\mathbb{Q})$ is representable by a rational algebraic cycle of codimension k.*

Conjecture 1.2.6 (Generalized Hodge Conjecture, Grothendieck [Gro69])**.** *Let X be a complex smooth projective variety. Suppose $L \subset H^k(X,\mathbb{Q})$ is a rational sub-Hodge structure of Hodge coniveau $\geq c$. Then there exists a closed algebraic subset $Z \subset X$ of codimension c such that L vanishes under the restriction map $H^k(X,\mathbb{Q}) \to H^k(X \setminus Z,\mathbb{Q})$, where $U := X \setminus Z$.*

Conjecture 1.2.5 is the special case of Conjecture 1.2.6 when the coniveau c is half the weight of the Hodge structure. The significance and recent advancements in the Hodge conjecture and the generalized Hodge conjecture are detailed in [Voi16II]. The following conjecture is stated by in [Voi13] by Voisin who also noticed that it is in fact a consequence of the Lefschetz standard conjecture (see [Voi14, Remark 2.30]).

Conjecture 1.2.7 (Voisin [Voi13])**.** *Let X be a smooth complex projective variety and $Y \subset X$ be a closed algebraic subset. Suppose $Z \subset X$ is a codimension k algebraic cycle, and assume that the cohomology class $[Z] \in H^{2k}(X,\mathbb{Q})$ vanishes in $H^{2k}(X \setminus Y,\mathbb{Q})$. Then there exists a codimension k cycle Z' on X with $\mathbb{Q}$-coefficients, which is supported on Y and such that $[Z'] = [Z]$ in $H^{2k}(X,\mathbb{Q})$.*

Combining the generalized Hodge conjecture (Conjecture 1.2.6) and the Bloch-Beilinson conjecture (Conjecture 1.1.14), the following conjecture is expected.

Conjecture 1.2.8. *Let $Z \subset CH^n(X \times X)$ be a self-correspondence of a smooth projective variety X of dimension n. If $[Z]^*|_{H^{i,0}(X)} = 0$ for some i, then $Z_*|_{Gr_F^i CH_0(X)} = 0$ for the same i as well, where $F^\bullet$ is the Bloch-Beilinson filtration and $Gr_F^\bullet$ is the graded part.*

Indeed, By the Künneth decomposition theorem and the Poincaré duality, $[Z] \in H^{2n}(X \times X,\mathbb{Q})$ can be identified as a graded map $[Z]^* : H^*(X,\mathbb{Q}) \to H^*(X,\mathbb{Q})$ that preserves Hodge structure. Let us fix a polarization on X and let $\phi : H^*(X,\mathbb{Q}) \to H^*(X,\mathbb{Q})$ be the restriction of $[Z]^*$ onto the subspace $N^1H^i(X,\mathbb{Q})$ which is the largest sub-Hodge structure of $H^i(X,\mathbb{Q})$

of coniveau at least 1. To be precise, $\phi(\alpha) := [Z]^*(\alpha)$ if $\alpha \in N^1H^i(X,\mathbb{Q})$ and $\phi(\alpha) = 0$ if $\alpha \in N^1H^i(X,\mathbb{Q})^\perp$ or if $\alpha \in H^k(X,\mathbb{Q})$ for $k \neq i$. The map ϕ preserves Hodge-structure, and when viewed as an element of $H^{2n}(X\times X,\mathbb{Q})$, is a Hodge class. By the Hodge conjecture, there is an algebraic cycle $Z_1 \in CH^n(X\times X)$ that represents $\phi \in H^{2n}(X\times X,\mathbb{Q})$. By the construction of Z_1, the algebraic cycle $Z_2 := Z - Z_1$ satisfies the condition $[Z_2]^*|_{H^i(X,\mathbb{Q})} = 0$. By the Bloch-Beilinson conjecture (Conjecture 1.1.14), we have $Z_{2*}|_{Gr^i_F CH_0(X)} = 0$. On the other hand, we have

Lemma 1.2.9. *Assuming the generalized Hodge conjecture (Conjecture 1.2.6) and the generalized Bloch-Beilinson conjecture (Conjecture 1.1.14), $Z_{1*}|_{Gr^k_F CH_0(X)} = 0$ for any k.*

Admitting Lemma 1.2.9 for the moment, we conclude $Z_* = Z_{1*} + Z_{2*}$ acts as 0 on $Gr^i_F CH_0(X)$, as desired.

Proof of Lemma 1.2.9. By construction, $[Z_1]^*|_{H^{k,0}(X)} = 0$ for any k. This implies that the sub-Hodge structure $[Z_1]^*H^k(X,\mathbb{Q}) \subset H^k(X,\mathbb{Q})$ has Hodge coniveau at least 1 for any k. By the generalized Hodge conjecture (Conjecture 1.2.6), there is an open dense subset $U \subset X$ such that $([Z_1]^*H^k(X,\mathbb{Q}))|_U = 0$ for any k. This means, by the Künneth decomposition theorem, that the cohomology class of $Z_1|_{U\times X}$ is 0. Therefore, by Conjecture 1.2.7 (which is implied by Conjecture 1.2.6), there is a cycle $Z' \in CH^n(X\times X)$, supported on $D\times X$, such that $[Z_1 - Z'] = 0 \in H^{2n}(X\times X,\mathbb{Q})$, where $D \subset X$ is the complement of U in X. Since Z' is supported on $D\times X$ and D is a proper closed subset of X, we have $Z'_* CH_0(X) = 0$. On the other hand, since $Z_1 - Z'$ is cohomologeous to 0, the (generalized) Bloch-Beilinson conjecture implies that $((Z_1 - Z'))_*|_{Gr^k_F CH^*(X)} = 0$ for any k. Taken together, we find that the action of $Z_1 = Z' + (Z_1 - Z')$ on $Gr^k_F CH_0(X)$ is 0, for any k, as desired. □

In the literature, Conjecture 1.2.8 is often called the generalized Bloch conjecture. It extends the classical Bloch conjecture, stated as follows in [Blo80] for surfaces.

Conjecture 1.2.10 (Bloch [Blo80])**.** *For a correspondence $Z \in CH^2(S\times T)_\mathbb{Q}$ between surfaces that induces a null map $[Z]^* : H^{2,0}(T) \to H^{2,0}(S)$, the induced morphism $Z_* : F^2CH_0(S) \to F^2CH_0(T)$ is identically zero. Here, $F^2CH_0(S)$ is defined as the kernel of the Albanese map from $CH_0(S)_{hom}$ to $Alb(S)$, and similarly for $F^2CH_0(T)$.*

If we take $Z = \Delta_X$, the diagonal of $X\times X$, Conjecture 1.2.8 predicts that if $H^{i,0}(X) = 0$ for all $i > 0$, then $CH_0(X)_{\mathbb{Q},hom} = 0$. This is part of the following conjecture, also named the generalized Bloch conjecture [Voi14, Conjecture 1.9].

Conjecture 1.2.11 (Generalized Bloch Conjecture)**.** *Let X be a smooth projective variety. Assume that $H^{p,q}(X) = 0$ for $p \neq q$ and $p < c$ (or $q < c$). Then the cycle class map*

$$cl : CH_i(X)_\mathbb{Q} \to H^{2m-2i}(X,\mathbb{Q})$$

is injective for $i \leq c-1$.

Some non-trivial incidences of Conjecture 1.2.11 are given in Chapter 4. Let $Y \subset \mathbb{P}^9$ be a general cubic eightfold. Its Fano variety of lines $F_1(Y)$ is a Fano manifold of dimension 12. It has been established that $H^{p,q}(F_1(Y)) = 0$ for $p \leq 1$ and $p \neq q$ [DM98], so Conjecture 1.2.11 predicts that $CH_i(F_1(Y))_{\mathbb{Q},hom} = 0$ for $i \leq 1$. In Chapter 4, we prove the following (see Theorem 4.3.6)

Theorem 1.2.12. *We have $CH_i(F_1(Y))_{\mathbb{Q},hom} = 0$ for any $i \leq 1$ and for any general cubic eightfold Y.*

1.2.2 The case of strict Calabi-Yau manifolds

Definition 1.2.13. A strict Calabi-Yau manifold X is a complex projective manifold of dimension at least 3 that is simply connected and has trivial canonical bundle, and such that for each $0 < i < \dim X$, there is no nonzero holomorphic forms of degree i on X.

Conjecture 1.2.8 takes the following form for strict Calabi-Yau manifolds, and more generally for smooth projective varieties X with $h^{i,0}(X) = 0$ for $0 < i < n = dimX$ and $h^{n,0}(X) = 1$.

Conjecture 1.2.14. *Let X be a strict Calabi-Yau manifold of dimension n. Let $\omega_X \in H^0(X, K_X)$ be a nowhere zero top degree holomorphic form on X. Let $Z \in CH^n(X \times X)_{\mathbb{Q}}$ be a self-correspondence such that $[Z]^*\omega = 0$. Then for any $z \in CH_0(X)_{\mathbb{Q},hom}$, we have $Z_* z = 0$.*

Indeed, since X is a strict Calabi-Yau manifold, $H^{k,0}(X) = 0$ for $0 < k < n$. Let $\Delta_X \in CH^n(X \times X)$ be the diagonal of $X \times X$. Then $[\Delta_X]^*|_{H^{k,0}(X)} = 0$ for $0 < k < n$. By Conjecture 1.2.8, $\Delta_{X*}|_{Gr_F^k CH_0(X)} = 0$ for $0 < k < n$, which implies that $Gr_F^k CH_0(X) = 0$ for $0 < k < n$. Therefore, the Bloch-Beilinson filtration on $CH_0(X)$ degenerates into

$$0 = F^{n+1}CH_0(X) \subset F^n CH_0(X) = \ldots = F^1 CH_0(X) = CH_0(X)_{hom} \subset F^0 CH_0(X) = CH_0(X).$$

By the assumption of Z, we have $[Z]^*|_{H^{n,0}(X)} = 0$. By Conjecture 1.2.8 again, we have $Z_*|_{Gr_F^n CH_0(X)} = 0$. Hence, Z_* acts as zero on $Gr_F^n CH_0(X) = F^n CH_0(X) = F^1 CH_0(X) = CH_0(X)_{hom}$, and we get the desired result.

1.2.3 Voisin's examples of Calabi-Yau manifolds

In Chapter 4, we aim to give evidence to Conjecture 1.2.14 for specific families of K-trivial varieties as constructed in [Voi04]. Let $Y \subset \mathbb{P}^n$ be a smooth cubic hypersurface of dimension $n-1$, and let $r \geq 0$ denote a nonnegative integer. Define $X = F_r(Y)$ as the Hilbert scheme that parametrizes the r-dimensional linear subspaces in Y. As proven in [Voi04, (4.41)], for $n+1 = \binom{r+3}{2}$ and a general Y, the variety X is a K-trivial variety of dimension $N = (r+1)(n-r) - \binom{r+3}{3}$. Specifically, when $r = 0$, X is an elliptic curve; and as established in [BD85], X is a hyper-Kähler manifold for $r = 1$. It is further shown (see Lemma 4.1.4) that for $r \geq 2$, X is a strict Calabi-Yau manifold. As is studied in Theorem 4.4.10 in Chapter 4, the small deformations of X is relatively easy to understand.

Theorem 1.2.15. *Assume $r \geq 2$. Let $X = F_r(Y)$ be the above strict Calabi-Yau manifold.*

(a) For any small deformation X' of X, there is a cubic hypersurface Y' such that $X' = F_r(Y')$.

(b) The dimension of deformation space of X is $\dim H^0(\mathbb{P}^n, \mathscr{O}_{\mathbb{P}^n}(3)) - \dim GL_{n+1}(\mathbb{C})$.

The distinct feature of the above manifolds $X = F_r(Y)$ among all K-trivial manifolds revolves around the presence of a self-rational map, $\Psi : X \dashrightarrow X$, referred to as the Voisin map. This map was introduced in [Voi04] through the following construction: Consider a general point $x \in X$, representing an r-dimensional linear space P_x within Y. As demonstrated in [Voi04, Lemma 8], there exists a unique $(r+1)$-dimensional linear subspace H_x

in $\mathbb{P}^n$ tangent to Y along P_x. The intersection $H_x \cap Y$ forms a cubic hypersurface containing P_x doubly, leaving a residual linear subspace in Y represented by a point $x' \in X$. This process defines the Voisin map as $\Psi(x) = x'$.

In Chapter 4, we prove the following fact about the Voisin maps in Theorem A:

Theorem 1.2.16. *Given a nowhere zero top degree holomorphic form $\omega \in H^0(X, K_X)$ on X, we have*

$$\Psi^* \omega = (-2)^{r+1} \omega.$$

By taking $Z = \Gamma_\Psi - (-2)^* \Delta_X \in CH(X \times X)$ where Γ_Ψ is the graph of Ψ and Δ_X is the diagonal of $X \times X$, it should be expected, by Conjecture 1.2.14, that for any $z \in CH_0(X)_{\mathbb{Q},hom}$, $\Psi_* z = (-2)^{r+1} z$. In the case $r = 2$, we succeed in proving this result in Theorem B in Chapter 4.

Theorem 1.2.17. *Let $Y \subset \mathbb{P}^9$ be a general cubic 8-fold. Let $X = F_2(Y)$ be the Fano variety of planes in Y and let $\Psi : X \dashrightarrow X$ be the Voisin map. Then for any $z \in CH_0(X)_{hom}$, we have*

$$\Psi_* z = -8z \text{ in } CH_0(X)_{\text{hom}}.$$

1.3 Some measures of irrationality

Let X be a projective variety over $\mathbb{C}$ of dimension n. By applying a general linear projection to $X \subset \mathbb{P}^N$, there exists a generically finite dominant rational map $\phi : X \dashrightarrow \mathbb{P}^n$. The following invariant, the degree of irrationality, was proposed and studied in [BDELU17].

Definition 1.3.1 ([BDELU17])**.** The degree of irrationality of X, denoted as $\mathrm{Irr}(X)$, is the minimal degree of dominant rational maps $\phi : X \dashrightarrow \mathbb{P}^n$. In other words,

$$\mathrm{Irr}(X) := \min\{\deg \phi : \text{there is a dominant rational map } \phi : X \dashrightarrow \mathbb{P}^n\}.$$

The degree of irrationality measures how far a variety is from being rational, in the sense that X is rational if and only if $\mathrm{Irr}(X) = 1$. If X is a smooth proper curve, the degree of irrationality of X is the gonality of the curve X, which is a classical invariant of a curve.

In their seminal work [BDELU17], Bastianelli, De Poi, Ein, Lazarsfeld, Ullery propose the following conjecture on the degree of irrationality of K3 surfaces.

Conjecture 1.3.2 ([BDELU17])**.** *Let $\{(S_d, L_d)\}_{d \in \mathbb{N}}$ be very general polarized K3 surfaces such that $L_d^2 = 2d - 2$. Then*

$$\limsup_{d \to \infty} \mathrm{Irr}(S_d) = +\infty.$$

In an attempt to determine the degree of irrationality and to better understand the geometry of the variety, the following two measures of irrationality are proposed and studied by Voisin in [Voi21].

Definition 1.3.3 ([Voi21])**.** (i) The fibering gonality of X, $\mathrm{Fibgon}(X)$, is the minimal number c such that there exists a rational dominant map $\pi : X \dashrightarrow B$ such that $\dim B = \dim X - 1$ and the general fiber is a connected curve C of gonality c.
(ii) The fibering genus of X, $\mathrm{Fibgen}(X)$, is the minimal number c such that there exists a rational dominant map $\pi : X \dashrightarrow B$ such that $\dim B = \dim X - 1$ and the general fiber is a connected curve C of geometric genus c.

The fibering genus of K3 surfaces has already been studied by Ein and Lazarsfeld in [EL20], under the name *Konno invariant*. They give a good estimate of the asymptotic behavior of fibering genus of K3 surfaces as the genus tends to infinity. Here the genus d of a K3 surface S is defined by the formula $2d-2=c_1(L)^2$, where L generates $\mathrm{Pic}(S)$.

Theorem 1.3.4 (Ein–Lazarsfeld [EL20])**.** *Let S_d be a polarized K3 surface of Picard rank* 1 *and genus d. Then*

$$\mathrm{Fibgen}(S_d)=\Theta(\sqrt{d}),$$

which means that there are constants C_1, $C_2>0$ such that $C_2\sqrt{d}<\mathrm{Fibgen}(S_d)<C_1\sqrt{d}$.

Corollary 1.3.5. *We have*

$$\lim_{d\to\infty}\mathrm{Fibgen}(S_d)=+\infty.$$

It is elementary to prove that for any smooth projective variety X, the following inequlities are satisfied.
(i) $\mathrm{Fibgon}(X)\leq\mathrm{Irr}(X)$.
(ii) $\mathrm{Fibgon}(X)\leq\frac{1}{2}(\mathrm{Fibgen}(X)-1)+2$.

In Chapter 3, we get a finer comparison for very general projective K3 surfaces (see Theorem 3.1.14).

Theorem 1.3.6. *Let S be a projective K3 surface whose Picard number is* 1*. Then one of the following two cases holds:*
(a) $\mathrm{Irr}(S)=\mathrm{Fibgon}(S)$*;*
(b) $\mathrm{Fibgen}(S)^2\leq\mathrm{Fibgon}(S)^{21}$.

Corollary 1.3.7. *Let $\{S_d\}_{d\in\mathbb{N}}$ be projective K3 surfaces such that the Picard group of S_d is generated by a line bundle with self intersection number $2d-2$. Then*

$$\limsup_{d\to\infty}\mathrm{Irr}(S_d)=+\infty\iff\limsup_{d\to\infty}\mathrm{Fibgon}(S_d)=+\infty.$$

Conjecture 1.3.2 predicts that $\limsup_d\mathrm{Irr}(S_d)=+\infty$, so it should be expected that $\limsup_d\mathrm{Fibgon}(S_d)=+\infty$ as well.

1.4 Hyper-Kähler Manifolds

Definition 1.4.1. A hyper-Kähler manifold X is defined as a compact Kähler manifold that is simply connected and for which $H^0(X,\Omega_X^2)$ is generated by an everywhere non-degenerate holomorphic 2-form σ_X.

The presence of an everywhere non-degenerate holomorphic 2-form implies that X has an even dimension. Hereafter, we denote by $m=2n$ the dimension of X and by σ_X a non-degenerate holomorphic 2-form of X (determined up to a scalar). Notably, a K3 surface is defined as a hyper-Kähler manifold of dimension 2. The broader aspects of hyper-Kähler varieties are extensively discussed in [Bea83, Fuj87, Huy99].

1.4.1 Beauville-Bogomolov-Fujiki Form

Hyper-Kähler manifolds exhibit a non-degenerate symmetric integral quadratic form on $H^2(X,\mathbb{Z})$. This form, which extends the Lefschetz intersection form applicable to K3 surfaces, has been discovered in [Bea83, Fuj87]. It is referred to as the Beauville-Bogomolov-Fujiki form.

Theorem 1.4.2 ([Bea83, Fuj87])**.** *Consider a hyper-Kähler manifold X of dimension $m = 2n$. There exist a unique integral quadratic form q_X on $H^2(X,\mathbb{Q})$ and a positive rational coefficient $\lambda_X \in \mathbb{Q}_{>0}$ such that:*
(1) for any $\alpha \in H^2(X,\mathbb{Q})$,

$$\int_X \alpha^m = \lambda_X q(\alpha)^n; \tag{1.1}$$

(2) q_X is indivisible, meaning that for any $k > 1$, q_X/k is not integral;
(3) $q_X(\alpha)$ is positive for any Kähler class α.

1.4.2 Lagrangian Subvarieties

Consider a hyper-Kähler manifold X of dimension $2n$ with a holomorphic 2-form σ_X.

Definition 1.4.3. A closed subvariety $L \subset X$ is called Lagrangian if:
(i) The restriction of σ_X to the smooth part L_{reg} of L vanishes, and
(ii) The dimension of L is n.

The deformation of *smooth* Lagrangian subvarieties with X has been investigated in [Voi92].

Theorem 1.4.4 (Voisin [Voi92])**.** *For a hyper-Kähler manifold X and a smooth Lagrangian subvariety $L \subset X$, let $\mathscr{X} \to \mathrm{Def}(X)$ represent the Kuranishi family of X, with $o \in \mathrm{Def}(X)$ as the reference point. Also, let $\mathrm{Def}(X,L)$ be the deformation germ of the pair (X,L). Then,*
(1) $\mathrm{Def}(X,L)$ is smooth, and both its fibers and the image of the natural map

$$\pi : \mathrm{Def}(X,L) \to \mathrm{Def}(X)$$

are also smooth.
(2) Furthermore,

$$\begin{aligned}\mathrm{Im}\pi &= \{t \in \mathrm{Def}(X) : [L] \text{ remains a Hodge class in } H^{2n}(\mathscr{X}_t,\mathbb{Q})\}\\ &= \{t \in \mathrm{Def}(X) : [\sigma_{X_t}] \in \ker(H^2(X,\mathbb{C}) \to H^2(L,\mathbb{C}))\}.\end{aligned}$$

The second equality in (2) implies that small deformations of X with constant Picard number contain a deformation of L. Theorem 1.4.4 crucially indicates that the deformation of a smooth Lagrangian subvariety within a hyper-Kähler manifold is unobstructed. It is used in Chapter 2 to justify the naturality of a certain technical condition.

1.4.3 Lagrangian fibrations

Consider a hyper-Kähler manifold X. A Lagrangian fibration of X is characterized as a holomorphic surjection $\pi : X \to B$, where the general fibers are Lagrangian subvarieties of X. Importantly, a Lagrangian fibration differs from a topological fibration in that it is a topological fibration solely over an open subset of the base and includes singular fibers.

This section describes the most important classical results on Lagrangian fibrations.

Theorem 1.4.5 (Matsushita [Mat99]). *Let $\pi : X \to B$ be a holomorphic surjection from a hyper-Kähler manifold X of dimension $2n$ to a compact complex variety B, assuming $0 < \dim B < 2n$ and the fibers are connected. Then $\dim B = n$ and $\pi : X \to B$ defines a Lagrangian fibration.*

Theorem 1.4.6 (Matsushita [Mat99], Voisin [Voi92]). *For a Lagrangian fibration $\pi : X \to B$, the general fibers are projective, hence are abelian varieties.*

Theorem 1.4.7 (Hwang [Hwa08]). *Given a Lagrangian fibration $\pi : X \to B$ with a smooth base B, then $B \cong \mathbb{P}^n$.*

Theorem 1.4.7 also holds under the assumption that X is merely smooth Kähler. This was demonstrated by Greb and Lehn [GL14] using a result on the deformation of Lagrangian fibrations by Matsushita [Mat16]. It is conjectured that the same holds when the base B is normal. Supporting evidence for this conjecture is provided in [Mat15], which discusses the intersection cohomology of the base. While this conjecture is straightforward in the context of K3 surfaces, it remains unresolved in a broader scope. In the case of four-dimensional hyper-Kähler manifolds, the conjecture has been proved by Huybrechts and Xu in [HX22].

1.4.4 Lagrangian families

A natural generalization of Lagrangian fibrations is the notion of Lagrangian families, introduced by Voisin in [Voi22I, Voi22II].

Definition 1.4.8 (Voisin). A Lagrangian family of a hyper-Kähler manifold X is a diagram

$$\begin{array}{ccc} \mathscr{L} & \xrightarrow{q} & X \\ \downarrow{\scriptstyle p} & & \\ B & & \end{array} \tag{1.2}$$

In this configuration, p is flat and projective, $\mathscr{L}$ and B are connected quasi-projective manifolds, and q maps the general fiber $L_b := p^{-1}(b)$, $b \in B$, birationally to a Lagrangian subvariety of X. For practical purposes, we will denote j_b as the composition $L_b \hookrightarrow \mathscr{L} \to X$.

As opposed to the Lagrangian fibrations, the existence of Lagrangian families are expected to be a less restrictive condition. In Chapter 2, we study the Abel-Jacobi map associated with the Lagrangian families. We briefly review Griffiths' theory on Abel-Jacobi maps in Section 2.1.1. In Chapter 2, we prove the following criterion to determine if the Abel-Jacobi map of a given Lagrangian family is trivial (see Proposition 2.1.14).

Theorem 1.4.9. *Consider a Lagrangian family of a hyper-Kähler manifold X of dimension $2n$, satisfying the following condition:*

> ♣ *For general $b \in B$, the contraction by $q^*\sigma_X$ gives an isomorphism $\lrcorner q^*\sigma_X : T_{B,b} \xrightarrow{\cong} H^0(L_b, \Omega_{L_b})$.*

Then the Abel-Jacobi map (2.3) is trivial if and only if for general $b \in B$, the restriction map

$$j_b^* : H^{2n-1}(X, \mathbb{Q}) \to H^{2n-1}(L_b, \mathbb{Q})$$

is zero.

This criterion gives a topological method to determine the triviality of the Abel-Jacobi maps of the Lagrangian families.

1.4.5 Mumford-Tate Groups

Our discussion closely follows the presentations of Mumford-Tate groups as found in [Del71, Section 7], [vGV16], and [Voi10, Section 4.2].

Consider a rational Hodge structure $H = (H_{\mathbb{Q}}, H^{p,q})_{p+q=r}$ of weight r. An algebraic group action, denoted μ, of $\mathbb{S}^1$ on $H_{\mathbb{R}}$ is defined as follows: For any $v \in H_{\mathbb{R}}$, with Hodge decomposition $v = \sum_{p+q=r} v^{p,q}$, and for any $z = e^{i\theta} \in \mathbb{S}^1 \subset \mathbb{C}^*$, we define

$$\mu(z).v := \sum_{p,q} z^p \bar{z}^q v^{p,q}.$$

Definition 1.4.10. The Mumford-Tate group $MT(H)$ for a Hodge structure H is defined as the smallest algebraic subgroup G of $GL(H_{\mathbb{Q}})$, defined over $\mathbb{Q}$, such that $G(\mathbb{R})$ contains $\mu(\mathbb{S}^1)$.

It should be noted that the Mumford-Tate group as defined here is referred to as the special Mumford-Tate group of H in [Voi10].

Returning to the context of hyper-Kähler manifolds, consider a projective hyper-Kähler manifold X with dimension $2n$, and let q denote the Beauville-Bogomolov-Fujiki form on $H^2(X,\mathbb{Q})$. The transcendental part of $H^2(X,\mathbb{Q})$, denoted by $H^2(X,\mathbb{Q})_{tr}$, is the minimal sub-Hodge structure containing $H^{2,0}$ and, thanks to the projectivity of X, can also be defined as the orthogonal complement to the Néron-Severi group $NS(X)_{\mathbb{Q}}$ in $H^2(X,\mathbb{Q})$ with respect to the Beauville-Bogomolov-Fujiki quadratic form. Due to the first Hodge-Riemann bilinear relation, the Hodge structure of $H^2(X,\mathbb{Q})$ is compatible with the Beauville-Bogomolov-Fujiki form q, which implies that the Mumford-Tate group $MT(H^2(X,\mathbb{Q})_{tr})$ is contained in $SO(H^2(X,\mathbb{Q})_{tr}, q)$. The following theorem shows that this inclusion is an equality for a very general polarized hyper-Kähler manifold (see, e.g. [vGV16, Lemma 9]):

Theorem 1.4.11. *Let X be a very general fiber of a complete family of lattice polarized deformations. Then the Mumford-Tate group $MT(H^2(X,\mathbb{Q})_{tr})$ is equal to $SO(H^2(X,\mathbb{Q})_{tr}, q)$.*

When X meet the criteria of Theorem 1.4.11, it is said that the Mumford-Tate group of X is maximal, or that X is a Mumford-Tate very general hyper-Kähler manifold. This is a technical condition that may help us to prove results for very general projective hyper-Kähler manifolds. Here are two examples of how the Mumford-Tate maximality condition appears in this thesis.

The first one gives a condition for the triviality of the Abel-Jacobi map for a Lagrangian family (see Theorem 2.1.15 and Proposition 2.1.16).

Theorem 1.4.12. *Consider a Lagrangian family on a Mumford-Tate very general hyper-Kähler manifold X of dimension $2n$ satisfying the following conditions*

(i) At least one of the Lagrangian subvarieties L_b in the Lagrangian family is smooth;

(ii) $h^{1,0}(L_b) \leq 2^{\lfloor \frac{b_2(X)_{tr}-3}{2} \rfloor}$.

Then the Abel-Jacobi map associated with this Lagrangian family is trivial.

The second example consists of the lower bound of the fibering genus of a projective hyper-Kähler manifold obtained by Voisin [Voi21].

Theorem 1.4.13 (Voisin [Voi21]). *Let X be a Mumford-Tate very general hyper-Kähler manifold of dimension $2n$ with $n \geq 3$ and $b_{2,\mathrm{tr}}(X) \geq 5$. Then*

$$\mathrm{Fibgen}(X) \geq \min\{n+2, 2^{\lfloor \frac{b_{2,\mathrm{tr}}(X)-3}{2} \rfloor}\}.$$

This result is further improved as follows in Chapter 3 (see Theorem 3.1.7).

Theorem 1.4.14. *Let X be a Mumford-Tate very general hyper-Kähler manifold of dimension $2n$ and assume $b_{2,\mathrm{tr}}(X) \geq 5$. Then*

$$\mathrm{Fibgen}(X) \geq \min\left\{n + \left\lceil \frac{-1+\sqrt{8n-7}}{2} \right\rceil, 2^{\lfloor \frac{b_{2,\mathrm{tr}}(X)-3}{2} \rfloor}\right\}.$$

1.5 Beauville's splitting conjecture and Voisin's conjectures

We follow closely the article [Voi16]. In this section, the Chow rings are considered with rational coefficients. Let X be a projective hyper-Kähler manifold. The splitting conjecture of Beauville [Bea07] predicts that the Bloch-Beilinson filtration on the Chow ring of X has a natural multiplicative splitting. As is explained in the Introduction of [Voi16], this implies the following conjecture, now often called the "weak splitting conjecture".

Conjecture 1.5.1 (Beauville's weak splitting conjecture [Bea07]). *Let X be a projective hyper-Kähler manifold. Then the cycle class map is injective on the subalgebra of $CH^*(X)$ generated by divisors.*

Beauville's weak splitting conjecture has been enlarged in [Voi08] as follows.

Conjecture 1.5.2 (Voisin [Voi08]). *Let X be a projective hyper-Kähler manifold. Let C^* be the subalgebra of $CH^*(X)$ generated by divisors and Chern classes. Then the cycle class map is injective on C^*.*

It has been shown by Beauville [Bea07, Examples 1.7] that the analogies of Conjecture 4.1.8 and Conjecture 1.5.2 for strict Calabi-Yau manifolds are false in general. Beauville constructs in *loc. cit.* a strict Calabi-Yau threefold X such that the cycle-class map $cl : CH_1(X)_{\mathbb{Q}} \to H^4(X, \mathbb{Q})$ is *not* injective on the subgroup generated by intersections of divisors. However, it is expected that the analogy of Conjecture 1.5.2 to strict Calabi-Yau manifolds still holds true for 0-cycles.

Conjecture 1.5.3 (Voisin). *Let X be a strict Calabi-Yau manifold of dimension n. Let $C \subset CH_0(X)_{\mathbb{Q}}$ be the degree n part of the subring of $CH^*(X)$ generated by the intersections of divisors and of Chern classes of X. Then the cycle class map*

$$cl : CH_0(X)_{\mathbb{Q}} \to H^{2n}(X, \mathbb{Q})$$

is injective on C.

We give a conditional result on Conjecture 1.5.3 for the strict Calabi-Yau 11-folds $F_2(Y)$ constructed by Voisin [Voi04] (see Theorem C in Chapter 4) and studied in this thesis.

Theorem 1.5.4. *Let $X = F_2(Y)$ be the Fano variety of planes of a general cubic 8-fold $Y \subset \mathbb{P}^9$. Suppose the indeterminacy locus of the Voisin map is a constant cycle subvariety. Then X satisfies Conjecture 1.5.3.*

In [Voi16], a filtration, now often called the "Voisin's filtration", is proposed for the Chow group of 0-cycles of X. As is defined in *loc. cit.*, for $x \in X$, let $O_x \subset X$ be the rational equivalence orbit of x, namely, O_x is the set of points in X that are rationally equivalent to x. Standard argument [Voi16, p. 2] shows that O_x is a countable union of closed algebraic subsets of X, which makes it possible to define the dimension of O_x as the maximal dimension among the irreducible components of O_x.

Definition 1.5.5 (Voisin [Voi03])**.** We define $S_iX \subset X$ to be the set of points in X whose orbit under rational equivalence has dimension $\geq i$. The filtration $S_\bullet$ is then defined by letting $S_iCH_0(X)$ be the subgroup of $CH_0(X)$ generated by classes of points $x \in S_iX$.

Highly motivated by the theory developed in *loc. cit.*, Conjecture 1.5.2 is enlarged in [Voi16]. Let $2n$ be the dimension of X. In the case of $i = n$, the conjectures in [Voi16] predict that two *constant cycle* subvarieties of dimension n, upon sharing an identical cohomological class, ought to share the same Chow class as well. Motivated by this conjecture, in Chapter 2, we study the Chow classes of Lagrangian subvarieties, raising the question of whether two Lagrangian subvarieties within the same hyper-Kähler manifold, sharing identical cohomological classes, also possess the same Chow class. Given that a constant-cycle subvariety of dimension n is easily seen to be Lagrangian, see [Voi16, Theorem 0.7], our question emerges as a natural extension of Voisin's conjecture. In Chapter 2, we give negative answer to this question by presenting a counter-example, thereby uncovering the intricacies and challenges associated with Voisin's conjecture.

We construct our counter-examples within the framework of the generalized Kummer varieties as introduced in [Bea83]. Let us remind the construction. Consider an abelian surface A and its Hilbert scheme $A^{[n+1]}$ of length $n+1$ subschemes. The morphism alb : $A^{[n+1]} \to A$ results from combining the Hilbert-Chow morphism with the summation map as follows:

$$A^{[n+1]} \to A^{(n+1)} \to A.$$

It is important to note that alb constitutes an isotrivial fibration. The generalized Kummer variety, denoted as $K_n(A)$, is defined as the fiber of alb over $0 \in A$. As demonstrated in [Bea83], $K_n(A)$ is a hyper-Kähler manifold of dimension $2n$. We then proceed to construct Lagrangian subvarieties in $K_n(A)$. For any element $x \in A$, we identify a subvariety Z_x within $K_n(A)$ that consists of Artinian subschemes of A of length $n+1$, supported on x and $-nx$, with multiplicities n and 1, respectively. For any curve $C \subset A$, we define $Z_C = \bigcup_{x \in C} Z_x$. In Section 2.5, we establish the following theorem:

Theorem 1.5.6. *(a) The subvariety Z_C is a Lagrangian subvariety in $K_n(A)$.*

(b) For numerically equivalent very ample curves C and C' in A, Z_C and $Z_{C'}$ share the same cohomological class in $H^{2n}(K_n(A), \mathbb{Z})$ but possess distinct Chow classes in $CH^n(K_n(A))$.

Chapter 2

On Abel-Jacobi Maps of Lagrangian Families

We study in this chapter the cohomological properties of Lagrangian families on projective hyper-Kähler manifolds. First, we give a criterion for the vanishing of Abel-Jacobi maps of Lagrangian families. Using this criterion, we show that under a natural condition, if the moduli map for the fibers of the Lagrangian family is maximal, its Abel-Jacobi map is trivial. We also construct Lagrangian families on generalized Kummer varieties whose Abel-Jacobi map is not trivial, showing that our criterion is optimal.

This chapter presents the main result of [Bai23]. The structure is as follows. In Section 2.1, we give a panorama of known results in this area. Section 2.2, Section 2.3 and Section 2.4 are the content of the article [Bai23]. In Section 2.2, we prove Proposition 2.1.14. In Section 2.3, we construct a Lagrangian fibration structure on the relative Albanese variety and use it to prove Theorem 2.1.15. In Section 2.4, we discuss the condition on the maximality of the variation of Hodge structures. In Section 2.5, we construct Lagrangian families satisfying a maximal condition whose Abel-Jacobi map is nontrivial, showing that Theorem 2.1.15 is optimal.

2.1 Background

The primary objective of this chapter is to address the following question:

Question 2.1.1. Suppose X is a hyper-Kähler manifold of dimension $2n$, and L_1 and L_2 are two Lagrangian subvarieties, both sharing the same cohomological class in $H^{2n}(X,\mathbb{Z})$. Are L_1 and L_2 necessarily rationally equivalent to each other?

The example presented in Section 2.5 gives a negative answer to Question 2.1.1.

2.1.1 Abel-Jacobi Maps

We begin by revisiting the Abel–Jacobi invariant and the Abel–Jacobi map introduced by Griffiths [Gri68], following the framework outlined in [Voi03, Chapter 12]. Throughout this section, X is a compact Kähler manifold of dimension n.

Intermediate Jacobian

Definition 2.1.2 (Griffiths)**.** The k-th intermediate Jacobian, denoted $J^{2k-1}(X)$, is the complex torus defined as:

$$J^{2k-1}(X) = H^{2k-1}(X,\mathbb{C})/(F^k H^{2k-1}(X) \oplus H^{2k-1}(X,\mathbb{Z})_{\text{tf}}).$$

By applying Poincaré duality, it follows that:

$$J^{2k-1}(X) \cong F^{n-k}H^{2n-2k+1}(X,\mathbb{C})^*/H_{2n-2k+1}(X,\mathbb{Z})_{\text{tf}}.$$

Here the subscript "tf" signifies the "torsion free part" of the integral cohomology.

Example 2.1.3. (i) When $k=1$, the intermediate Jacobian $J^1(X) = \text{Pic}^0(X)$, representing the degree 0 part of the Picard group of X.
(ii) When $k=n$, the intermediate Jacobian $J^{2n-1}(X) = \text{Alb}(X)$ is the Albanese variety of X.

Abel-Jacobi Invariant

Let $\mathscr{Z}^k(X)$ denote the free abelian group generated by irreducible subvarieties of codimension k in X. The cycle class map,

$$cl : \mathscr{Z}^k(X) \to H^{2k}(X,\mathbb{Z})$$

maps the class of a subvariety of codimension k to its cohomological class in $H^{2k}(X,\mathbb{Z})$. Let $\mathscr{Z}^k(X)_{hom}$ represent the kernel of the cycle class map, where the subscript "hom" signifies "homologous to 0".

Associated with any k-cocycle $Z \in \mathscr{Z}^k(X)_{hom}$ that is homologous to 0 is an element $\alpha_Z \in J^{2k-1}(X)$, termed the *Abel-Jacobi invariant of Z*, in the intermediate Jacobian of the corresponding degree. The construction of α_Z can be briefly described as follows [Gri68], [Voi03, 12.1.2].

Given that Z is homologous to 0, there exists a chain Γ of codimension $2k-1$ such that $\partial\Gamma = Z$. The integration current along Γ,

$$\int_\Gamma : \omega \mapsto \int_\Gamma \omega$$

can be interpreted as an element in $F^{n-k}H^{2n-2k-1}(X,\mathbb{C})^*$. In fact, if ω and ω' yield the same cohomological class in $F^{n-k}H^{2n-2k-1}(X,\mathbb{C})$, then, by a deep result of Hodge theory, they differ by an exact form $d\phi$ with $\phi \in F^{n-k-1}\mathscr{A}_{\mathbb{C}}^{2n-2k-2}(X)$. By Stokes' theorem, $\int_\Gamma d\phi = \int_Z \phi$, which vanishes due to type reasons. Additionally, if Γ and Γ' are two chains such that $\partial\Gamma = \partial\Gamma' = Z$, then modulo the image of $H_{2n-2k+1}(X,\mathbb{Z})$ in $F^{n-k}H^{2n-2k+1}(X,\mathbb{C})^*$, the linear forms $\int_\Gamma$ and $\int_{\Gamma'}$ are the same. Hence, there exists a uniquely determined element:

$$\alpha_Z = \left[\int_\Gamma\right] \in F^{n-k}H^{2n-2k-1}(X,\mathbb{C})^*/H_{2n-2k+1}(X,\mathbb{Z}) = J^{2n-2k+1}(X).$$

Proposition 2.1.4 ([Voi03], Lemme 21.19)**.** *Let Z be an algebraic cycle that is rationally equivalent to 0 in X. Then the Abel–Jacobi invariant of Z is 0.*

By Proposition 2.1.4, we are now equipped with a map

$$\begin{array}{rccc} \Phi_X^k : & CH^n(X)_{hom} & \to & J^{2n-1}(X) \\ & Z = \partial\Gamma & \mapsto & \int_\Gamma \end{array}$$

Abel-Jacobi Map of a Family

Let $\mathscr{Z} \subset B \times X$ be a flat family of subvarieties of codimension k in X. Specifically, B is a connected complex manifold, and $\mathscr{Z}$ is a subvariety of codimension k in $B \times X$, flat over B. Denote by $p : \mathscr{Z} \to B$ and $q : \mathscr{Z} \to X$ the projection maps to the two components. Let $0 \in B$ be a reference point.

Definition 2.1.5. The Abel–Jacobi map of the family $\mathscr{Z} \subset B \times X$ with respect to the reference point $0 \in B$ is the map

$$\begin{array}{rccc} \Psi^{AJ}_{\mathscr{Z}} : & B & \to & J^{2k-1}(X) \\ & b & \mapsto & \Phi^k_X(q_* p^*(b) - q_* p^*(0)). \end{array}$$

Theorem 2.1.6 (Griffiths [Gri68])**.** *(i) The Abel–Jacobi map $\Psi^{AJ}_{\mathscr{Z}} : B \to J^{2k-1}(X)$ is holomorphic.*

(ii) The image of the differential of $\Psi^{AJ}_{\mathscr{Z}}$ at any point lies in $H^{k-1,k}(X) \subset H^{2k-1}(X, \mathbb{C})$.

2.1.2 Lagrangian Families

Consider X as a projective hyper-Kähler manifold of dimension $2n$. Voisin introduced the notion of Lagrangian families in her work [Voi22I, Voi22II], which serves as a generalizations of Lagrangian fibrations.

Definition 2.1.7 (Voisin [Voi22I, Voi22II])**.** A Lagrangian family of a hyper-Kähler manifold X is a diagram

$$\begin{array}{ccc} \mathscr{L} & \xrightarrow{q} & X \\ \downarrow{\scriptstyle p} & & \\ B & & \end{array} \tag{2.1}$$

In this configuration, p is flat and projective, $\mathscr{L}$ and B are connected quasi-projective manifolds, and q maps the general fiber $L_b := p^{-1}(b)$, $b \in B$, birationally to a Lagrangian subvariety of X. For practical purposes, we will denote by j_b the composition $L_b \hookrightarrow \mathscr{L} \to X$.

Let us give some examples of Lagrangian families documented in the literature:

Example 2.1.8. (i) Any Lagrangian fibration of X naturally leads to a Lagrangian family. Note that the existence of a Lagrangian fibration necessitates that X has a Picard number of at least 2, whereas a very general projective hyper-Kähler manifold typically has a Picard number of 1.

(ii) (Voisin [Voi92]) Let S be a K3 surface and C be a curve in S. The curve C can move in a family $\{C_b\}_{b \in U}$ where U is an open subset of the Hilbert scheme of deformations of C in S. This constitutes a Lagrangian family of S. Moreover, consider $S^{[n]}$, the Hilbert scheme of n points of S, which is a hyper-Kähler manifold of dimension $2n$. Then $\{C^{(n)}_b\}_{b \in U}$ is a Lagrangian family of $S^{[n]}$.

(iii) (Voisin [Voi92]) Let $Y \subset \mathbb{P}^5$ be a smooth cubic fourfold, and $H \subset \mathbb{P}^5$ be a general hyperplane. Then the Fano surface $F_1(Y \cap H)$ of lines of $Y \cap H$, is a Lagrangian subvariety of the Fano variety $F_1(Y)$ of lines of Y, which is a hyper-Kähler fourfold by [BD85]. The general hyperplanes $H \subset \mathbb{P}^5$ are parameterized by an open subset U of $(\mathbb{P}^5)^*$. The family $\{F_1(Y \cap H)\}_{H \in U}$ provides a Lagrangian family of $F_1(Y)$.

(iv) (Iliev-Manivel [IM08]) Let us consider a linear inclusion $\mathbb{P}^5 \subset \mathbb{P}^6$ and a smooth cubic fourfold $Y \subset \mathbb{P}^5$. Now, let $Z \subset \mathbb{P}^6$ be a general smooth cubic fivefold containing Y. Let $S_Z := F_2(Z)$ denote the Fano surface of planes in Z. The map

$$\begin{array}{rccc} j_Z: & S_Z & \to & F_1(Y) \\ & P & \mapsto & P \cap H \end{array}$$

is generically 1:1 on its image and the image is a Lagrangian subvariety. Consequently, $\{S_Z\}_{Z \in U}$, where U is an open subset of all cubic fivefolds containing Y, provides a Lagrangian family of $F_1(Y)$.

2.1.3 Voisin's Conjectures Revisited

Let X be a projective hyper-Kähler manifold of dimension $2n$. Recall the definition of Voisin's filtration [Voi16] previously mentioned in definition 1.5.5. The subset S_nX contains the points whose rational equivalence orbit has a dimension of n.

Proposition 2.1.9 ([Voi16], Theorem 0.7)**.** *An irreducible subvariety of dimension n, denoted as $L \subset X$, is a constant-cycle subvariety if and only if $L \subset S_nX$.*

Based on this proposition, the following conjecture is posed in [Voi16]:

Conjecture 2.1.10 ([Voi16])**.** *Let L and L' be two n-dimensional* constant cycle *subvarieties of a hyper-Kähler manifold X. If the cohomological classes $[L] = [L']$ in $H^{2n}(X, \mathbb{Q})$, then L is rationally equivalent to L' as algebraic cycles in X.*

It is established [Voi16, Theorem 0.7] that an n-dimensional constant cycle subvariety L of X is Lagrangian. Question 2.1.1 is the question whether, in Conjecture 2.1.10, the condition "constant cycle subvarieties" can be substituted with "Lagrangian subvarieties".

Lagrangian families serve as an essential source of Lagrangian subvarieties sharing the same cohomological class. In view of Question 2.1.1, we are motived to study the following

Problem 2.1.11. *Consider a Lagrangian family on a hyper-Kähler manifold X of dimension $2n$ given by a diagram as in (2.1). What can be said of the map*

$$\begin{array}{rccc} \psi_{\mathscr{L}}: & B & \to & CH^n(X) \\ & b & \mapsto & q_*(L_b) \end{array} \quad ? \tag{2.2}$$

The condition constant cycle is a strong condition for Lagrangian subvarieties. Notice that in contrast with Lagrangian subvarieties, constant cycle Lagrangian subvarieties cannot deform into families.

Lemma 2.1.12. *Small deformations of constant cycle Lagrangian subvarieties of X are no longer constant cycle subvarieties.*

Proof. Following the notations in [Voi16], let

$$S_nX := \{x \in X : \text{ the rational equivalence orbit of } x \text{ has dimension} \geq n\}.$$

As is shown in [Voi16, Theorem 1.3], S_nX is a countable union of irreducible varieties of dimension $\leq n$ and constant cycle Lagrangian subvarieties of X are exactly irreducible components of S_nX of dimension n. Hence, constant cycle Lagrangian subvarieties of X are rigid. □

As is described in Section 2.1, a weaker invariant of algebraic cycles in a projective manifold is the Abel-Jacobi invariant. Problem 2.1.11 thus motivates the following question.

Problem 2.1.13. *Consider a Lagrangian family of a hyper-Kähler manifold X of dimension $2n$ given by a diagram as in (2.1). Let $0 \in B$ be a point. Under which conditions is the Abel-Jacobi map*

$$\begin{array}{rccc} \Psi^{AJ}_{\mathscr{L}}: & B & \to & J^{2n-1}(X) \\ & b & \mapsto & \Phi^n_X(q_*(L_b - L_0)) \end{array} \tag{2.3}$$

trivial?

In many instances, Lagrangian families provide affirmative answers to Problem 2.1.11, as highlighted in all examples of Lagrangian families in Example 2.1.8. However, as detailed in Section 2.5, an explicit example has been constructed that offers a negative solution to Problem 2.1.13. This example is derived from a Lagrangian family of the generalized Kummer varieties. The advantage of considering a Lagrangian family over a mere pair of Lagrangian subvarieties is the utilization of the differential theory of families developed by Griffiths [Gri68] to explore the triviality of the Abel-Jacobi map of such a family. Consequently, we have developed a cohomological criterion in Section 2.2 to determine if a given Lagrangian family possesses a trivial Abel-Jacobi map. Additionally, an explicit Lagrangian family of generalized Kummer varieties is constructed and shown to exhibit a nontrivial Abel-Jacobi map using the criterion we have developed. This implies that any two Lagrangian subvarieties lacking the same Abel-Jacobi invariant in this family provide a negative response to Question 2.1.1.

It is important to note that this example does not present counter-examples to Voisin's Conjecture 2.1.10. In fact, it adds depth and intricacy to Voisin's Conjecture, making it even more engaging and nuanced.

2.1.4 Organization of the chapter

In this chapter, we first give a criterion for the vanishing of the Abel-Jacobi map (2.3) for Lagrangian families of a hyper-Kähler manifold (see also Proposition 2.2.1).

Proposition 2.1.14. *Consider a Lagrangian family on a hyper-Kähler manifold X of dimension $2n$ as in (2.1), satisfying the following condition :*

> ♣ *For general $b \in B$, the contraction by $q^*\sigma_X$ gives an isomorphism $\lrcorner q^*\sigma_X : T_{B,b} \xrightarrow{\cong} H^0(L_b, \Omega_{L_b})$.*

Then the Abel-Jacobi map (2.3) is trivial if and only if for general $b \in B$, the restriction map

$$j_b^* : H^{2n-1}(X, \mathbb{Q}) \to H^{2n-1}(L_b, \mathbb{Q})$$

is zero.

The condition ♣ is natural. According to [Voi92, Proposition 2.4], the deformations of a *smooth* Lagrangian subvariety are non-obstructed, and a local deformation is still Lagrangian. Therefore, if we take (B,b) to be a germ of the Hilbert scheme of deformations of a *smooth* Lagrangian subvariety $L \subset X$, and $\mathscr{L} \to B$ the corresponding family, then condition ♣ holds since $T_{B,b} \cong H^0(L_b, N_{L_b/X})$ by unobstructedness and $\lrcorner\sigma_X : H^0(L_b, N_{L_b/X}) \to H^0(L_b, \Omega_{L_b})$ is an isomorphism for a smooth Lagrangian variety.

Using this criterion, we give a response to Problem 2.1.13.

Theorem 2.1.15. *Consider a Lagrangian family on a hyper-Kähler manifold X of dimension $2n$ given by a diagram as in (2.1), satisfying condition ♣. Assume that the variation of Hodge structures on the degree 1 cohomology of the fibers of $p : \mathscr{L} \to B$ is maximal, i.e., the period map*

$$\begin{array}{rccc} \mathscr{P}: & B & \to & Gr(h^{1,0}(L), H^1(L,\mathbb{C})) \\ & b & \mapsto & H^{1,0}(L_b) \subset H^1(L_b,\mathbb{C}) \cong H^1(L,\mathbb{C}), \end{array} \tag{2.4}$$

where L is a general fiber of $p : \mathscr{L} \to B$, is generically a local immersion. Then the Abel-Jacobi map (2.3) is trivial.

This response to Problem 2.1.13 is conditional. However, it can be shown that the condition "maximal variation of Hodge structures" cannot be dropped. In fact, we construct in Section 2.5 Lagrangian families satisfying ♣ for which the Abel-Jacobi map is shown to be nontrivial using Proposition 2.1.14. The variation of weight 1 Hodge structures of the constructed Lagrangian families is not maximal.

In Section 2.4, we shall explore under which conditions the variation of weight 1 Hodge structures is maximal. Let $H^2(X,\mathbb{Q})_{tr}$ be the orthogonal complement of $NS(X)_{\mathbb{Q}}$ in $H^2(X,\mathbb{Q})$ with respect to the Beauville-Bogomolov-Fujiki form q of X (see [Bea83]) and let $b_2(X)_{tr}$ be the dimension of $H^2(X,\mathbb{Q})_{tr}$ .We prove the following result (see also Proposition 2.4.2):

Proposition 2.1.16. *Consider a Lagrangian family on a hyper-Kähler manifold X of dimension $2n$ given by a diagram as in (2.1), satisfying condition ♣. Assume that the Mumford-Tate group of the Hodge structure $H^2(X,\mathbb{Q})$ is maximal, i.e. it is the special orthogonal group of $(H^2(X,\mathbb{Q})_{tr}, q)$, and assume that $b_2(X)_{tr} \geq 5$. If $h^{1,0}(L_b)$ is smaller than $2^{\lfloor \frac{b_2(X)_{tr}-3}{2} \rfloor}$, then the variation of weight 1 Hodge structures of p is maximal.*

Corollary 2.1.17. *Under the same assumptions as in Proposition 2.1.16, the Abel-Jacobi map (2.3) is trivial.*

Let $p : \mathscr{L} \to B$, $q : \mathscr{L} \to X$ be a Lagrangian family. Up to shrinking B, we may assume that the map $p : \mathscr{L} \to B$ is smooth. Let

$$\pi : \mathscr{A} := Alb(\mathscr{L}/B) \to B$$

be the relative Albanese variety of $p : \mathscr{L} \to B$. In the proof of Theorem 2.1.15 and Proposition 2.1.16, we use a similar construction to those in [LSV17, Voi22I] to get a holomorphic 2-form $\sigma_{\mathscr{A}}$ on $\mathscr{A}$. If we assume the condition ♣, $\pi : \mathscr{A} \to B$ is a Lagrangian fibration with respect to $\sigma_{\mathscr{A}}$(see Section 2.3). It is interesting to notice that, by this construction, under condition ♣, we can translate the problem concerning Lagrangian families to a problem concerning Lagrangian fibrations. However, the total space of the Lagrangian fibration is no longer a hyper-Kähler manifold, but a completely integrable system over an open subset of the base, as studied for instance in [DM98, Chapter 7].

2.2 A criterion

In this section, we establish a criterion for the vanishing of the Abel-Jacobi map (2.3). Let X be a hyper-Kähler manifold of dimension $2n$. With the notation $j_b : L_b \to X$ as in the introduction, we prove

Proposition 2.2.1. *Consider a Lagrangian family of hyper-Kähler manifold X of dimension $2n$ given by a diagram as in (2.1).*
(a) If for general $b \in B$, the restriction map

$$j_b^* : H^{2n-1}(X, \mathbb{Q}) \to H^{2n-1}(L_b, \mathbb{Q})$$

is zero, then the Abel-Jacobi map (2.3) is trivial.
(b) If condition ♣ holds (see Proposition 2.1.14), then the converse of (a) holds.

Remark 2.2.2. The cohomology group $H^{2n-1}(L_b, \mathbb{Q})$ has a Hodge structure of weight $2n-1$ and level 1. By Hodge symmetry and using the fact $j_b^* : H^{2n-1}(X, \mathbb{Q}) \to H^{2n-1}(L_b, \mathbb{Q})$ is a morphism of Hodge structures, we conclude that $j_b^* : H^{2n-1}(X, \mathbb{Q}) \to H^{2n-1}(L_b, \mathbb{Q})$ is zero, if and only if $j_b^* : H^{n-1,n}(X) \to H^{n-1,n}(L_b)$ is zero.

Proof. Let $d\Psi^{AJ}_{\mathscr{L},b} : T_{B,b} \to H^n(X, \Omega_X^{n+1})$ denote the differential of the Abel-Jacobi map $\Psi^{AJ}_{\mathscr{L}}$ at point $b \in B$ (Theorem 2.1.6). Let $j_{b*} : H^0(L_b, \Omega_{L_b}) \to H^n(X, \Omega_X^{n+1})$ be the Gysin map, which is the Serre dual to the following composition

$$j_b^* : H^n(X, \Omega_X^{n-1}) \to H^n(L_b, \Omega^{n-1}_{\mathscr{L}|L_b}) \to H^n(L_b, \Omega^{n-1}_{L_b}). \tag{2.5}$$

By the above remark, the proposition follows from the following lemma and the fact that

$$\cup \sigma_X : H^n(X, \Omega_X^{n-1}) \to H^n(X, \Omega_X^{n+1})$$

is an isomorphism since $\wedge \sigma_X : \Omega_X^{n-1} \to \Omega_X^{n+1}$ is a vector bundle isomorphism. □

Lemma 2.2.3. *The following diagram is commutative:*

$$\begin{array}{ccc} T_{B,b} & \xrightarrow{d\Psi^{AJ}_{\mathscr{L},b}} & H^n(X, \Omega_X^{n-1}) \\ \downarrow{\scriptstyle \lrcorner q^* \sigma_X} & & \downarrow{\scriptstyle \cup \sigma_X} \\ H^0(L_b, \Omega_{L_b}) & \xrightarrow{j_{b*}} & H^n(X, \Omega_X^{n+1}). \end{array} \tag{2.6}$$

Proof. We are going to show that the Serre dual of the diagram (2.6) is commutative.

Let $L^\bullet \Omega^i_{\mathscr{L}|L_b}$ be the Leray filtration [Voi03, Chapter 16] induced on the vector bundle $\Omega^i_{\mathscr{L}|L_b}$ by the exact sequence

$$0 \to p^* \Omega_{B,b} \to \Omega_{\mathscr{L}|L_b} \to \Omega_{L_b} \to 0,$$

and defined by $L^j \Omega^i_{\mathscr{L}|L_b} = p^* \Omega^j_{B,b} \wedge \Omega^{i-j}_{\mathscr{L}|L_b}$. Since L_b is supposed to be Lagrangian, $q^* \sigma_X \in H^0(L_b, L^1 \Omega^2_{\mathscr{L}|L_b})$ and thus the cup product

$$\cup q^* \sigma_X : \Omega^\bullet_{\mathscr{L}|L_b} \to \Omega^{\bullet+2}_{\mathscr{L}|L_b}$$

sends $L^k \Omega^\bullet_{\mathscr{L}|L_b}$ to $L^{k+1} \Omega^{\bullet+2}_{\mathscr{L}|L_b}$. Denoting $\overline{q^* \sigma_X}$ the image of $q^* \sigma_X$ in $H^0(L_b, Gr^1_L \Omega^2_{\mathscr{L}|L_b}) \cong H^0(L_b, \Omega_{L_b}) \otimes \Omega_{B,b}$, this implies the existence of the following commutative diagram

$$\begin{array}{ccc} L^1 \Omega^{n+1}_{\mathscr{L}|L_b} = \Omega^{n+1}_{\mathscr{L}|L_b} & \longrightarrow & K_{L_b} \otimes p^* \Omega_{B,b} = Gr^1_L \Omega^{n+1}_{\mathscr{L}|L_b} \\ {\scriptstyle \cup q^* \sigma_X} \uparrow & & \uparrow {\scriptstyle \cup \overline{q^* \sigma_X}} \\ L^0 \Omega^{n-1}_{\mathscr{L}|L_b} = \Omega^{n-1}_{\mathscr{L}|L_b} & \longrightarrow & \Omega^{n-1}_{L_b} = Gr^0_L \Omega^{n-1}_{\mathscr{L}|L_b}, \end{array} \tag{2.7}$$

where K_{L_b} is the canonical bundle of L_b. Taking the n-th cohomology of (2.7) and combine it with $q^*: H^n(X,\Omega_X^\bullet) \to H^n(L_b,\Omega^\bullet_{\mathscr{L}|L_b})$, we get the following commutative ladder

$$\begin{array}{ccccc} H^n(X,\Omega_X^{n+1}) & \xrightarrow{q^*} & H^n(L_b,\Omega^{n+1}_{\mathscr{L}|L_b}) & \longrightarrow & H^n(L_b,K_{L_b}\otimes p^*\Omega_{B,b}) \cong \Omega_{B,b} \\ \cup\sigma_X \uparrow & & \cup q^*\sigma_X \uparrow & & \cup\overline{q^*\sigma_X} \uparrow \\ H^n(X,\Omega_X^{n-1}) & \xrightarrow{q^*} & H^n(L_b,\Omega^{n-1}_{\mathscr{L}|L_b}) & \longrightarrow & H^n(L_b,\Omega^{n-1}_{L_b}). \end{array} \tag{2.8}$$

Lemma 2.2.4. *The composite in the first row of the diagram (2.8) coincides with the dual of $d\Psi_{\mathscr{L},b}$.*

Proof. Let $\bar{p}:\bar{\mathscr{L}}\to\bar{B}$, $\bar{q}:\bar{\mathscr{L}}\to X$ be a relative completion of $p:\mathscr{L}\to B$ with respect to the morphism $q:\mathscr{L}\to X$. More precisely, $\bar{B}$ is a smooth projective variety, $\bar{p}$ is a flat morphism extending p, and $\bar{q}$ is a morphism extending q. The extended family has a Abel–Jacobi map $\Psi^{AJ}_{\bar{\mathscr{L}}}:\bar{B}\to J^{2n-1}(X)$ that induces a morphism of complex tori $\psi:\mathrm{Alb}(\bar{B})\to J^{2n-1}(X)$ whose differential is given by the morphism of Hodge structures ([Voi03, Théorème 12.17])

$$\bar{q}_*\bar{p}^*: H^{2d-1}(\bar{B},\mathbb{Z})\to H^{2n-1}(X,\mathbb{Z}),$$

where $d=\dim B$. It is well-known that the differential of the Albanese map $\mathrm{alb}:\bar{B}\to\mathrm{Alb}(\bar{B})$ at a point $b\in B$ is given by the dual of the evaluation map $ev_b: H^0(\bar{B},\Omega_{\bar{B}})\to\Omega_{\bar{B},b}$. Hence, the dual of $d\Psi^{AJ}_{\mathscr{L},b}$ is given by the correspondance $\bar{p}_*\bar{q}^*: H^n(X,\Omega_X^{n+1})\to H^0(\bar{B},\Omega_{\bar{B}})$ composed with the evaluation map $ev_b: H^0(\bar{B},\Omega_{\bar{B}})\to\Omega_{\bar{B},b}\cong\Omega_{B,b}$. The domain $\bar{B}$ can be restricted to B before the evaluation map $ev_b: H^0(\bar{B},\Omega_{\bar{B}})\to\Omega_{B,b}$. Therefore, the dual of $d\Psi^{AJ}_{\mathscr{L},b}$ is given by the correspondance $p_*q^*: H^n(X,\Omega_X^{n+1})\to H^0(B,\Omega_B)$ composed with the evaluation map $ev_b: H^0(B,\Omega_B)\to\Omega_{B,b}$. The Gysin map $p_*: H^n(\mathscr{L},\Omega^{n+1}_{\mathscr{L}})\to H^0(B,\Omega_B)$ is given by the Leray filtration. Taken together, the dual of $d\Psi^{AJ}_{\mathscr{L},b}$ is given by

$$H^n(X,\Omega_X^{n+1})\xrightarrow{q^*}H^n(\mathscr{L},\Omega^{n+1}_{\mathscr{L}})\to H^n(\mathscr{L},K_{\mathscr{L}/B}\otimes p^*\Omega_B)\to H^0(B,R^np_*\Omega_{\mathscr{L}/B}\otimes\Omega_B)\xrightarrow{ev_b}\Omega_{B,b}.$$

Since the restriction to the fiber L_b and taking the Leray filtration are commutative processes, the above composite of maps is equal to the one that takes the restriction to the fiber L_b first and then takes the Leray filtration. The latter one is exactly the first row of the diagram (2.8). □

As in (2.5), the composite in the second row is j_b^*. Taking into account of Lemma 2.2.4, the diagram 2.8 is indeed the Serre dual of the diagram (2.6). This concludes the proof of Lemma 2.6. □

2.3 Lagrangian fibrations

In this section, we associate to a Lagrangian family satisfying condition ♣ on X a fibration on another variety, which turns out to be holomorphic symplectic in such a way that the fibration is lagrangian. This is constructed with the help of a construction appeared in [LSV17, Voi22I]. We use this Lagrangian fibration to prove Theorem 2.1.15.

General Constructions

Let (2.1) be a Lagrangian family of a hyper-Kähler manifold X of dimension $2n$. We fix a relative polarization of $\mathscr{L} \to B$ given by a hyperplane section of X. Let

$$\pi : \mathscr{A} := Alb(\mathscr{L}/B) \to B$$

be the relative Albanese variety of $p : \mathscr{L} \to B$.

Lemma 2.3.1. *There exist an open dense subset $B_0 \subset B$ and a finite covering $B_0' \to B_0$ such that, denoting $p_0' : \mathscr{L}_0' \to B_0'$ the base change of p under $B_0' \to B_0 \hookrightarrow B$ and $\pi_0' : \mathscr{A}_0' \to B_0'$ the relative Albanese variety of p_0', there is a cycle $Z_0 \in CH^n(\mathscr{A}_0' \times_{B_0'} \mathscr{L}_0')$ such that*

$$[Z_0]^* : p_{0*}'\Omega_{\mathscr{L}_0'/B_0'} \to \pi_{0*}'\Omega_{\mathscr{A}_0'/B_0'}$$

is an isomorphism.

Proof. Let $B_0 \subset B$ be the subset of regular points of $p : \mathscr{L} \to B$. For $b \in B_0$, let $j : C_b \hookrightarrow L_b$ be a complete intersection curve and J_{C_b} the Jacobian variety of C_b. By Lefschetz theorem on hyperplane sections, $j_* : J_{C_b} \to A_b := Alb(L_b)$ is surjective. By the semi-simplicity of polarized Hodge structures, there exists a $\mathbb{Q}$-section $s : A_b \to J_{C_b}$ of j_*, i.e., there exists $N > 0$ such that $j_* \circ s = N \cdot id_{A_b}$. On $J_{C_b} \times C_b$, we have the Poincaré divisor $d_b \in CH^1(J_{C_b} \times C_b)$ such that $[d_b]^* : H^1(C_b, \mathbb{Q}) \to H^1(J_{C_b}, \mathbb{Q})$ is an isomorphism of Hodge structures, exhibiting the inverse of the pull-back of the Jacobi map $jac_b : C_b \to J_{C_b}$. Let us consider

$$\begin{array}{ccc} A_b \times C_b & \xrightarrow{(s, id_{C_b})} & J_{C_b} \times C_b \\ \downarrow{\scriptstyle (id_{A_b}, j)} & & \\ A_b \times L_b & & \end{array}$$

and define $Z_b := (id_{A_b}, j)_*(s, id_{C_b})^*(d_b) \in CH^n(A_b \times L_b)$. Then $[Z_b]^* : H^1(L_b, \mathbb{Q}) \to H^1(A_b, \mathbb{Q})$ is given by the composition

$$H^1(L_b, \mathbb{Q}) \xrightarrow{j^*} H^1(C_b, \mathbb{Q}) \xrightarrow{d_b^*} H^1(J_{C_b}, \mathbb{Q}) \xrightarrow{s^*} H^1(A_b, \mathbb{Q}), \tag{2.9}$$

which is an isomorphism by the definition of s. In fact, let $(j_*)^* : H^1(A_b, \mathbb{Q}) \to H^1(J_{C_b}, \mathbb{Q})$ be the pull-back map of $j_* : J_{C_b} \to A_b = Alb(L_b)$. Precomposing the left-hand-side of $(j_*)^* : H^1(A_b, \mathbb{Q}) \to H^1(J_{C_b}, \mathbb{Q})$ with the natural identification $H^1(Alb(L_b), \mathbb{Q}) \cong H^1(L_b, \mathbb{Q})$, and post-composing the right hand side with the pull-back of the Jacobi map $jac_b^* : H^1(J_{C_b}, \mathbb{Q}) \to H^1(C_b, \mathbb{Q})$, we would get the pull-back map $j^* : H^1(L_b, \mathbb{Q}) \to H^1(C_b, \mathbb{Q})$. Since $jac_b^* : H^1(J_{C_b}, \mathbb{Q}) \to H^1(C_b, \mathbb{Q})$ is the inverse of $d_b^* : H^1(C_b, \mathbb{Q}) \to H^1(J_{C_b}, \mathbb{Q})$, the composite $d_b^* \circ j^* : H^1(L_b, \mathbb{Q}) \to H^1(J_{C_b}, \mathbb{Q})$ of the first two maps of the composition (2.9) is exactly the pull-back map of $j_* : J_{C_b} \to A_b = Alb(L_b)$ at the level of cohomology, after the natural identification $H^1(Alb(L_b), \mathbb{Q}) \cong H^1(L_b, \mathbb{Q})$. Since $j_* \circ s = N \cdot id_{A_b}$, we get the desired isomorphism.

The cycles Z_b are defined fiberwise, but standard arguments [Voi14, Chapter 3] show that they can be constructed in family over a smooth generically finite cover $B_0' \to B_0$. Let us spell out the standard arguments. By the theory of Hilbert schemes, there are countably many connected projective B_0-schemes $H_1, \ldots, H_i, \ldots$ together with the universal families of cycles $\mathscr{Z}_1, \ldots, \mathscr{Z}_i, \ldots$ such that for each $i \in \mathbb{N}$, every fiber of $\mathscr{Z}_i \to H_i$ is a cycle

$Z_b \in \mathscr{Z}^1(A_b \times L_b)$ such that $[Z_b]^* : H^1(L_b, \mathbb{Q}) \to H^1(A_b, \mathbb{Q})$ is an isomorphism. By the construction of the previous paragraph, the structure map $\bigcup_{i\in\mathbb{N}} H_i \to B_0$ is surjective, thus there is $i \in \mathbb{N}$, such that $H_i \to B_0$ is surjective. Since $H_i \to B_0$ is projective, we may take a multisection of the map $B_0' \to B_0$, and the universal family $\mathscr{Z}_i$ restricted to B_0 gives the desired 1-cycle. □

For the sake of simplicity, we shall note B_0, $\mathscr{L}_0$ and $\mathscr{A}_0$ instead of B_0', $\mathscr{L}_0'$ and $\mathscr{A}_0'$. We define a holomorphic 2-form $\sigma_{\mathscr{A}_0}$ on $\mathscr{A}_0$ by setting

$$\sigma_{\mathscr{A}_0} := [Z_0]^* q_0^* \sigma_X, \tag{2.10}$$

where $q_0 : \mathscr{L}_0 \to X$ is the natural map.

Proposition 2.3.2. *(a) The 2-form $\sigma_{\mathscr{A}_0}$ is closed.*
(b) $\sigma_{\mathscr{A}_0}$ vanishes on fibers of $\pi_0 : \mathscr{A}_0 \to B_0$.
(c) The composite morphism $\kappa : T_{B_0} \xrightarrow{\lrcorner q_0^*\sigma_X} p_{0*}\Omega_{\mathscr{L}_0/B_0} \xrightarrow{[Z_0]^*} \pi_{0*}\Omega_{\mathscr{A}_0/B_0}$ is given by the contraction $\lrcorner\sigma_{\mathscr{A}_0}$.

Proof. (a) Let $Z_q := (id, q_0)_* Z_0 \in CH(\mathscr{A}_0 \times X)$. Then by the projection formula, $\sigma_{\mathscr{A}_0} = [Z_q]^*\sigma_X$. Let $\mathscr{A}'$ be a projective completion of $\mathscr{A}_0$. Then Z_q extends to a cycle $\bar{Z}_q$ of $\mathscr{A}' \times X$. $\sigma_{\mathscr{A}_0}$ extends to a 2-form $\sigma_{\mathscr{A}'} := [\bar{Z}_q]^*\sigma_X$ which is automatically closed since $\mathscr{A}'$ is projective. Thus, $\sigma_{\mathscr{A}_0} = \sigma_{\mathscr{A}'|\mathscr{A}_0}$ is also closed.

(b) Since Z_0 is a cycle in $\mathscr{A}_0 \times_{B_0} \mathscr{L}_0 \subset \mathscr{A}_0 \times \mathscr{L}_0$, the morphism $[Z_0]^* : H^*(\mathscr{L}_0) \to H^*(\mathscr{A}_0)$ preserves the Leray filtrations on both sides. Therefore, $\sigma_{\mathscr{A}_0} \in H^0(\mathscr{A}_0, \pi_0^*\Omega_{B_0} \wedge \Omega_{\mathscr{A}_0}) \subset H^0(\mathscr{A}_0, \Omega^2_{\mathscr{A}_0})$ since $q_0^*\sigma_X \in H^0(\mathscr{L}_0, p_0^*\Omega_{B_0} \wedge \Omega_{\mathscr{L}_0})$ by the definition of Lagrangian families. Therefore, $\sigma_{\mathscr{A}_0}$ vanishes on the fibers of $\pi_0 : \mathscr{A}_0 \to B_0$.

(c) By Lemma 2.3.1, $[Z_0]^*$ induces an isomorphism $H^0(B_0, \Omega_{B_0} \otimes p_{0*}\Omega_{\mathscr{L}_0/B_0}) \to H^0(B_0, \Omega_{B_0} \otimes \pi_{0*}\Omega_{\mathscr{A}_0/B_0})$ which sends $\lrcorner q_0^*\sigma_X$ to $\lrcorner\sigma_{\mathscr{A}_0}$. □

By (b) and (c) of the above proposition, we get the following diagram that is commutative up to a sign:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & T_{\mathscr{A}_0/B_0} & \longrightarrow & T_{\mathscr{A}_0} & \longrightarrow & \pi_0^*T_{B_0} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle (\pi_0^*\kappa)^*} & & \downarrow{\scriptstyle \lrcorner\sigma_{\mathscr{A}_0}} & & \downarrow{\scriptstyle \pi_0^*\kappa} & & \\ 0 & \longrightarrow & \pi_0^*\Omega_{B_0} & \longrightarrow & \Omega_{\mathscr{A}_0} & \longrightarrow & \Omega_{\mathscr{A}_0/B_0} & \longrightarrow & 0. \end{array} \tag{2.11}$$

Here, the commutativity of the second square is dual to Proposition 2.3.2 (c). Since the dual of $\lrcorner\sigma_{\mathscr{A}_0} : T_{\mathscr{A}_0} \to \Omega_{\mathscr{A}_0}$ is given by $-\lrcorner\sigma_{\mathscr{A}_0} : T_{\mathscr{A}_0} \to \Omega_{\mathscr{A}_0}$, the first square is anti-commutative.

Lemma 2.3.3. *If condition ♣ (see Proposition 2.1.14) holds for all $b \in B_0$, then $\sigma_{\mathscr{A}_0}$ is nowhere degenerate on $\mathscr{A}_0$.*

Proof. If condition ♣ holds, then $\pi_0^*\kappa : \pi_0^*T_{B_0} \to \Omega_{\mathscr{A}_0/B_0}$ is an isomorphism. By the commutativity of (2.11) and the five lemma, $\lrcorner\sigma_{\mathscr{A}_0} : T_{\mathscr{A}_0} \to \Omega_{\mathscr{A}_0}$ is an isomorphism, which means that $\sigma_{\mathscr{A}_0}$ is nowhere degenerate. □

Symmetry

Let (2.1) be a Lagrangian family of a hyper-Kähler manifold X of dimension $2n$. We fix a relative polarization of $\mathscr{L} \to B$ given by a hyperplane section of X. Let $b \in B$ be a general point. The infinitesimal variation of Hodge structures on degree 1 cohomology of the fibers of $p : \mathscr{L} \to B$ at b is given by (see [Voi03, Lemme 10.19])

$$\bar{\nabla} : T_{B,b} \to Hom(H^0(L_b, \Omega_{L_b}), H^1(L_b, \mathscr{O}_{L_b})).$$

Precomposed with the map $\lrcorner q^*\sigma_X : T_{B,b} \to H^0(L_b, \Omega_{L_b})$, the map $\bar{\nabla}$ induces a bilinear map

$$\begin{array}{rccc} S: & T_{B,b} \times T_{B,b} & \to & H^1(L_b, \mathscr{O}_{L_b}) \\ & (u,v) & \mapsto & \bar{\nabla}_u(v \lrcorner q^*\sigma_X). \end{array} \tag{2.12}$$

Proposition 2.3.4. *The bilinear map S is symmetric in the sense that $S(u,v) = S(v,u)$ for any $u, v \in T_{b,B}$.*

Proof. By Griffiths' transversality [Voi03, Chapter 17], $S(u,v) = \rho(u) \lrcorner (v \lrcorner q^*\sigma_X)$, where $\rho : T_{B,b} \to H^1(L_b, T_{L_b})$ is the Kodaira-Spencer map. Therefore, we need to show that the following diagram is commutative

$$\begin{array}{ccc} T_{B,b} \otimes T_{B,b} & \xrightarrow{\rho \otimes id} & H^1(L_b, T_{L_b}) \otimes T_{B,b} \\ \downarrow{\scriptstyle \lrcorner q^*\sigma_X \otimes id} & & \downarrow{\scriptstyle \beta} \\ H^0(L_b, \Omega_{L_b}) \otimes T_{B,b} & \xrightarrow{\alpha} & H^1(L_b, \mathscr{O}_{L_b}), \end{array} \tag{2.13}$$

where $\alpha(\omega, v) = \rho(v) \lrcorner \omega$ and $\beta(u, \chi) = u \lrcorner (\chi \lrcorner q^*\sigma_X)$.

To see the commutativity of (2.13), restrict the commutative ladder (2.11) to L_b and apply the cohomology on L_b, then the commutativity of (2.11) implies the commutativity of (2.13). Indeed, (2.13) is the connecting homomorphism of the cohomology of (2.11) tensored by $T_{B,b}$. □

Remark 2.3.5. When condition ♣ is satisfied, the symmetry of S comes from the completely integrable system structure on $(\mathscr{A}_0, \sigma_{\mathscr{A}_0})$. What we proved is in fact the symmetry of

$$\begin{array}{rccc} S': & T_{B,b} \times T_{B,b} & \to & H^1(A_b, \mathscr{O}_{A_b}) \\ & (u,v) & \mapsto & \bar{\nabla}_u(v \lrcorner \sigma_{\mathscr{A}_0}). \end{array} \tag{2.14}$$

Fixing a relative polarisation on $\mathscr{A}_0 \to B_0$, we have natural isomorphisms (always under ♣): $H^1(A_b, \mathscr{O}_{A_b}) \cong H^0(A_b, \Omega_{A_b})^* \cong T^*_{B,b}$ and we can thus view S' as an element in $T^*_{B,b} \otimes T^*_{B,b} \otimes T^*_{B,b}$. If this relative polarisation is principal, Donagi and Markman proved in [DM98, Lemma 7.2] that S' lies in $Sym^3 T^*_{B,b}$. This result is called "weak cubic condition" in [DM98, Lemma 7.2]. See also [Voi17, Theorem 4.4]

Now we are ready to prove the main theorem.

Proof of Theorem 2.1.15. Under the assumptions of Theorem 2.1.15, assume by contradiction that the Abel–Jacobi map (2.3) is not constant. In what follows, we fix a relative polarization on $p_0 : \mathscr{L}_0 \to B_0$ induced from a hyperplane section of X, so that $R^{2n-1}p_{0*}\mathbb{Q} \cong R^1 p_{0*}\mathbb{Q}$. By Proposition 2.1.14, the morphism

$$j^* : H^{2n-1}(X, \mathbb{Q}) \to R^{2n-1}p_{0*}\mathbb{Q} \cong R^1 p_{0*}\mathbb{Q}$$

of variations of Hodge structures on an open subset $B_0 \subset B$ containing b is not zero. Hence, there is a non-zero locally constant sub-variation of Hodge structures $I := \mathrm{Im} j^* \subset R^1 p_{0*}\mathbb{Q}$. Since I is locally constant, for any $\omega \in I_b^{1,0}$ and $u \in T_{B,b}$, $\nabla_u(\omega) = 0$. Recall that ♣ means that $\lrcorner q^* \sigma_X : T_{B,b} \to H^0(L_b, \Omega_{L_b})$ is bijective. Let $F := (\lrcorner q^* \sigma_X)^{-1}(I^{1,0}) \subset T_{B,b}$. Then by the symmetry of S given by Proposition 2.3.4, F lies in the kernel of $\bar{\nabla}$, which contradicts our assuption that the variation of Hodge structures is maximal.

□

2.4 Maximal variations

In this section, we study under what conditions could the variation of Hodge structures of a Lagrangian family be maximal. Consider a Lagrangian family of a hyper-Kähler manifold X of dimension $2n$ satisfying the condition ♣ given by the diagram as in (2.1). Let $U \subset B$ be a simply connected open subset of $B_0 \subset B$ and let

$$\begin{array}{rccc} \mathscr{P}: & U & \to & Gr(h^{1,0}(L), H^2(L,\mathbb{C})) \\ & b & \mapsto & H^{1,0}(L_b) \subset H^1(L_b,\mathbb{C}) \cong H^1(L,\mathbb{C}), \end{array} \tag{2.15}$$

be the local period map of the Lagrangian family.

In what follows, we are going to use the universal property of the Kuga-Satake construction proved in [vGV16, Proposition 6].

Theorem 2.4.1 ([vGV16])**.** *Let (H^2, q) be a polarized Hodge structure of hyper-Kähler type of dimension ≥ 5. Assume that the Mumford–Tate group of the Hodge structure on H^2 is maximal, namely the special orthogonal group of (H^2, q). Let H be a simple effective weight-1 Hodge structure, such that there exists an injective morphism of Hodge structures of bidegree $(-1,-1)$*

$$H^2 \hookrightarrow Hom(H, A)$$

for some weight-1 Hodge structure A. Then H is a direct summand of the Kuga–Satake Hodge structure $H^1_{KS}(H^2, q)$. In particular, $\dim H \geq 2^{\lfloor \frac{\dim H^2 - 1}{2} \rfloor}$.

With the same notations as in the introduction, we prove

Proposition 2.4.2. *Assume that the Mumford-Tate group of the Hodge structure $H^2(X,\mathbb{Q})$ is maximal, i.e. it is the special orthogonal group of $(H^2(X,\mathbb{Q})_{tr}, q)$ and assume $b_2(X)_{tr} \geq 5$. If the dimension of $H^{0,1}(L_b)$ is smaller than $2^{\lfloor \frac{b_2(X)_{tr}-3}{2} \rfloor}$ for a general fiber L_b of $p : \mathscr{L} \to B$, then the variation of weight 1 Hodge structure of p is maximal.*

Proof. We use the same argument as in [vGV16] where similar results were proved for Lagrangian fibrations. Assuming that the period map (2.15) is not generically an immersion, we are going to prove that $\dim H^{0,1}(L_b) \geq 2^{\lfloor \frac{b_2(X)_{tr}-3}{2} \rfloor}$. By assumption, the nonempty general fibers of $\mathscr{P}$ are of dimension ≥ 1. Let $b \in U$ be a general point and let B_b the fiber of $\mathscr{P}$ passing through b. Let $U_b = B_b \cap U$. Then the fibers of $\pi|_{U_b} : \mathscr{A}_{U_b} \to U_b$ are isomorphic with each other. Thus, up to a base change by a finite covering of U_b, we may assume $\pi|_{U_b} : \mathscr{A}_{U_b} \to U_b$ is trivial, i.e., $\mathscr{A}_{U_b} = U_b \times A_b$. Let $\pi_{F_b} : \mathscr{A}_{F_b} \to F_b$ be a smooth completion of $\pi|_{U_b}$, then $\mathscr{A}_{F_b}$ is birational to $F_b \times A_b$, which gives a morphism $H^2(\mathscr{A}_{F_b}) \to H^2(F_b \times A_b)$. Recall by Lemma 2.3.1, we get a morphism $[Z]^* : H^2(X) \to H^2(\mathscr{A})$ that sends σ_X to a

holomorphic 2-form which is non-degenerate on $\mathscr{A}_U$. Finally, the rational map $\mathscr{A}_{F_b} \dashrightarrow \mathscr{A}$ induces $H^2(\mathscr{A}) \to H^2(\mathscr{A}_{F_b})$. Compositing all these maps, we get a morphism

$$\alpha : H^2(X)_{tr} \hookrightarrow H^2(X) \to H^2(\mathscr{A}) \to H^2(\mathscr{A}_{F_b}) \to H^2(F_b \times A_b) \to H^1(F_b) \otimes H^1(A_b), \quad (2.16)$$

where the last map is given by the projection in the Künneth decomposition.

Lemma 2.4.3. *$\alpha : H^2(X)_{tr} \to H^1(F_b) \otimes H^1(A_b)$ is injective.*

Proof. Since $h^{2,0}(X) = 1$ and $H^{2,0}(X)$ is orthogonal to $NS(X)$ with respect to the Beauville-Bogomolov-Fujiki form, $H^2(X)_{tr}$ is a simple Hodge structure. Therefore, to show the injectivity of α it suffices to show that α is not zero. We claim that $\alpha(\sigma_X) \neq 0$. Indeed, Since A_b is Lagrangian with respect to $\sigma_{\mathscr{A}}$ (Proposition 2.3.2 (b)), in the Künneth's decomposition of $H^2(A_b \times F_b)$, the image of σ_X in $H^0(F_b) \otimes H^2(A_b)$ is zero. If furthermore $\alpha(\sigma_X) = 0$ in $H^1(F_b) \otimes H^1(A_b)$, then the image of σ_X on $F_b \times A_b$ comes from a 2-form on F_b, which has rank $\leq \dim F_b$. Therefore, the rank of $\sigma_{\mathscr{A}}$ has rank $\leq \dim F_b$ on $\mathscr{A}_{U_b}$. On the other hand, the codimension of $\mathscr{A}_{U_b}$ in $\mathscr{A}_U$ is $\dim B - \dim F_b$, and thus the non-degeneration of $\sigma_{\mathscr{A}_U}$ implies that $\sigma_{\mathscr{A}}$ has rank $\geq 2\dim F_b$ on $\mathscr{A}_{U_b}$. This is a contradiction since we are assuming $\dim F_b \geq 1$. □

We are now in the position to use the universal property of the Kuga-Satake construction (see Theorem 2.4.1 above). Since $\alpha : H^2(X)_{tr} \to H^1(F_b) \otimes H^1(A_b)$ is nonzero, there is at least one simple direct factor A of A_b such that $H^2(X)_{tr} \to H^1(F_b) \otimes H^1(A)$ is nonzero thus injective. Taking H^2 as $H^2(X)_{tr}$, we conclude by Theorem 2.4.1 that

$$\dim H^{0,1}(L_b) = \dim A_b \geq \dim A \geq \frac{1}{2} \times 2^{\lfloor \frac{\dim H^2(X)_{tr}-1}{2} \rfloor} = 2^{\lfloor \frac{b_2(X)_{tr}-3}{2} \rfloor},$$

as desired. □

2.5 Example of a Lagrangian family with nontrivial Abel-Jacobi map

Recall the construction of generalized Kummer varieties introduced in [Bea83]. Let A be an abelian surface and $A^{[n+1]}$ the Hilbert scheme of length $n+1$ subschemes of A. Let $\mathrm{alb} : A^{[n+1]} \to A$ be the composition of the Hilbert-Chow morphism and the summation map

$$A^{[n+1]} \to A^{(n+1)} \to A.$$

Note that alb is an isotrivial fibration. The generalized Kummer variety $K_n(A)$ is defined to be the fiber of alb over $0 \in A$. As is shown in [Bea83], $K_n(A)$ is a hyper-Kähler manifold of dimension $2n$.

In this section, we are going to construct Lagrangian families of $X := K_n(A)$ for $n \geq 2$, satisfying condition ♣ and whose Abel-Jacobi map is *not* trivial.

For any $x \in A$, one defines a subvariety Z_x of $K_n(A)$ consisting of Artinian subschemes of A of length $n+1$ supported on x and $-nx$, with multiplicities n and 1, respectively. By [Bri77, Proposition VI.1.1], Z_x is a *rational* variety of dimension $n-1$ if x is *not* an $(n+1)$-torsion point. Let $Z = \bigcup_{x \in A} Z_x$ and let $\pi : Z \to A$ send elements in Z_x to x. For any curve $C \subset A$, define $Z_C = \bigcup_{x \in C} Z_x$.

Now let B be a connected open subset of the Hilbert scheme of deformations of a smooth curve $C \subset A$ and $\mathscr{C} \to B$ the corresponding family.

Lemma 2.5.1. *$\{Z_C\}_{C\in B}$ is a Lagrangian family of $K_n(A)$ satifying condition ♣.*

Proof. Since for general C, Z_C is a fibration over a curve C whose general fibers are rational, any holomorphic 2-form on Z_C is 0. Furthermore, $\dim Z_C = n = \dim K_n(A)/2$. These imply that $\{Z_C\}_{C\in B}$ is a Lagrangian family. We now show that this family satisfies condition ♣. Denoting $\mathscr{L}$ the total space of the family $\{Z_C\}_{C\in B}$ and L a general fiber, and using as before the following notation

$$\begin{array}{ccc} \mathscr{L} & \xrightarrow{q} & X \\ \downarrow{\scriptstyle p} & & \\ B & & \end{array},$$

we need to show that $\lrcorner q^*\sigma_{K_n(A)} : H^0(L,N_{L/Z}) = H^0(L,N_{L/\mathscr{L}}) \to H^0(L,\Omega_L)$ is an isomorphism. Since the general fibers of π are rational, $q^*\sigma_{K_n(A)} = \pi^*\sigma_A$, where σ_A is the unique (up to coefficients) holomorphic 2-form on A. Therefore, we can conclude by the commutativity of the following diagram

$$\begin{array}{ccc} H^0(C,N_{C/A}) & \xrightarrow{\lrcorner\sigma_A} & H^0(C,\Omega_C) \\ \downarrow{\scriptstyle \pi^*} & & \downarrow{\scriptstyle \pi^*} \\ H^0(L,N_{L/Z}) & \xrightarrow{\lrcorner\pi^*\sigma_A} & H^0(L,\Omega_L) \end{array}$$

noting that the two vertical arrows are isomorphims since the fibers of π are rational, and that $\lrcorner\sigma_A : H^0(C,N_{C/A}) \to H^0(C,\Omega_C)$ is an isomorphism since σ_A is nondegenerate. □

Proposition 2.5.2. *The Abel-Jacobi map of the Lagrangian family $\{Z_C\}_{C\in B}$ is* not *trivial.*

Proof. Let $i : C \hookrightarrow A$ be a general curve in the family $\mathscr{C} \to B$. By Proposition 2.2.1(a), it suffices to show that the restriction map $H^{2n-1}(X,\mathbb{Q}) \to H^{2n-1}(Z_C,\mathbb{Q})$ is nonzero.

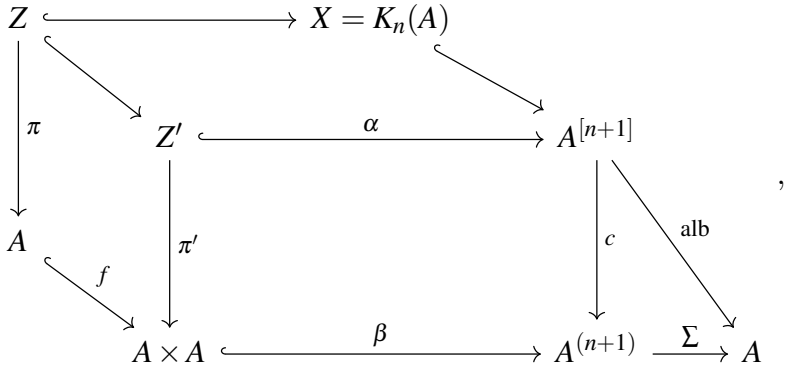
define an injective morphism

$$\begin{array}{rccc} \beta: & A\times A & \hookrightarrow & A^{(n+1)} \\ & (x,y) & \mapsto & n\{x\}+\{y\}, \end{array}$$

where we use the notation $\{x\} \in \mathscr{Z}_0(A)$ the 0-cycle of the point $x \in A$. Consider the following pull-back diagram definitionning a subvariety $Z' \subset A^{[n+1]}$

$$\begin{array}{ccc} Z' & \xhookrightarrow{\alpha} & A^{[n+1]} \\ \downarrow{\scriptstyle \pi'} & & \downarrow{\scriptstyle c} \\ A\times A & \xhookrightarrow{\beta} & A^{(n+1)} \end{array},$$

where $c : A^{[n+1]} \to A^{(n+1)}$ is the Hilbert-Chow morphism. Then $Z = Z' \cap K_n(A) \subset A^{[n+1]}$. We have the following commutative diagram where all three squares are pull-back diagrams



Here $f : A \to A \times A$ defined by $x \mapsto (x, -nx)$ is the fiber over $0 \in A$ of the trivial fibration $\sum \circ \beta : A \times A \to A$.

By [dCM02, Corollary 5.1.5], $[Z']^* : H^{2n-1}(A^{[n+1]}, \mathbb{Q}) \to H^1(A \times A, \mathbb{Q})$ is surjective. Furthermore, the restriction map $f^* : H^1(A \times A, \mathbb{Q}) \to H^1(A, \mathbb{Q})$ is surjective since f is the fiber of a trivial fibration. These imply that $[Z]^* : H^{2n-1}(X, \mathbb{Q}) \to H^1(A, \mathbb{Q})$ is surjective. Finally, since the restriction map $i^* : H^1(A, \mathbb{Q}) \to H^1(C, \mathbb{Q})$ is injective by Lefschetz hyperplane theorem, the composition map $i^* \circ [Z]^* : H^{2n-1}(X, \mathbb{Q}) \to H^1(C, \mathbb{Q})$ is nonzero. This implies that the restriction map $H^{2n-1}(X, \mathbb{Q}) \to H^{2n-1}(Z_C, \mathbb{Q})$ is nonzero, as desired. □

Chapter 3

On Some Birational Invariants of Hyper-Kähler Manifolds

We study in this chapter three birational invariants of projective varieties, the degree of irrationality, the fibering gonality and the fibering genus. We first improve the lower bound in a recent result of Voisin bounding from below the fibering genus of a Mumford-Tate very general projective hyper-Kähler manifold by a constant depending on its dimension and the second Betti number. We also compare the relations between these birational invariants for projective $K3$ surfaces of Picard number 1 and study the asymptotic behaviors of their degree of irrationality and fibering gonality.

This chapter presents the main result of [Bai24]. The structure is as follows. In Section 3.1, we give a panorama of known results in this area and a brief introduction to the main results of this chapter. Section 3.2, Section 3.3 and Section 3.4 are the content of the article [Bai24]. In Section 3.2, we prove Theorem 3.1.7. In Section 3.3, we prove Theorem 3.1.14. In Section 3.4, we prove Propositions 3.1.15 and 3.1.16. In a final Section 3.5, we present some related results and questions.

3.1 Background and Introduction

3.1.1 Some measures of irrationality

Let X be a projective variety over $\mathbb{C}$ of dimension n. The following invariant, known as the *degree of irrationality*, was initially defined and studied by Hisao Yoshihara and others before being further explored in [BDELU17].

Definition 3.1.1. The *degree of irrationality* of X, denoted as $\mathrm{Irr}(X)$, is the minimal degree of dominant rational maps $\varphi : X \dashrightarrow \mathbb{P}^n$. Formally,

$$\mathrm{Irr}(X) := \min\{\deg \varphi : \text{there is a dominant rational map } \varphi : X \dashrightarrow \mathbb{P}^n\}.$$

In their important work, Bastianelli, De Poi, Ein, Lazarsfeld, and Ullery studied $\mathrm{Irr}(X)$ for very general hypersurfaces X of large degree in [BDELU17]. Note that the following Theorem C from [BDELU17], regarding the degree of irrationality for hypersurfaces, has also appeared in [BCD] and was proven earlier in the case of surfacees in the thesis by Renza Cortini.

Theorem 3.1.2 (=Theorem C in [BDELU17]). *Let $X \subset \mathbb{P}^{n+1}$ be a very general smooth hypersurface of dimension n and degree $d \geq 2n+1$. Then* $\mathrm{Irr}(X) = d-1$. *Furthermore, if $d \geq 2n+2$, then any rational mapping $f : X \dashrightarrow \mathbb{P}^n$ with* $\deg(f) = d-1$ *is given by the projection from a point of X.*

In an attempt to determine the degree of irrationality and to better understand the geometry of the variety, the following two measures of irrationality are proposed and studied by Voisin in [Voi21].

Definition 3.1.3 ([Voi21]). (i) The fibering gonality of X, $\mathrm{Fibgon}(X)$, is the minimal number c such that there exists a rational dominant map $\pi : X \dashrightarrow B$ such that $\dim B = \dim X - 1$ and the general fiber is a connected curve C of gonality c.
(ii) The fibering genus of X, $\mathrm{Fibgen}(X)$, is the minimal number c such that there exists a rational dominant map $\pi : X \dashrightarrow B$ such that $\dim B = \dim X - 1$ and the general fiber is a connected curve C of geometric genus c.

Following [Voi21], the following inequalities hold between these two measures of irrationality and the degree of irrationality.

Lemma 3.1.4. *For any smooth projective variety X, we have*
(i) $\mathrm{Fibgon}(X) \leq \mathrm{Irr}(X)$.
(ii) $\mathrm{Fibgon}(X) \leq \frac{1}{2}(\mathrm{Fibgen}(X)-1)+2$.

3.1.2 Measures of irrationality of hyper-Kähler manifolds

Let X be a projective hyper-Kähler manifold of dimension $2n$. The following result is directly implied by the main result of [AP92].

Theorem 3.1.5 (Alzati-Pirola [AP92]). *The degree of irrationality of X is greater than n.*

A direct proof of Theorem 3.1.5 goes as follows.

Proof. Let $X \dashrightarrow Y$ be a dominant generically finite rational map of degree d to a smooth variety Y such that $H^{l,0}(Y) = 0$ for any $l > 0$. We are going to show that $d > n$. First we desingularize the rational map as $\phi : \tilde{X} \to Y$ and let $\sigma_{\tilde{X}}$ be the pull-back of the holomorphic 2-form of X on $\tilde{X}$.

Consider the fibered product $Z' = \tilde{X} \times_Y \tilde{X} \times_Y \ldots \times_Y \tilde{X} \subset \tilde{X} \times \ldots \times \tilde{X}$ with d copies of $\tilde{X}$. Let Δ_{ij} be the pull back of the diagonal under the natural projection map $pr_{ij} : Z' \to \tilde{X} \times \tilde{X}$. Let Z be the Zariski closure of $Z' - \bigcup_{i,j} \Delta_{ij}$ in $\tilde{X} \times \ldots \times \tilde{X}$. Since $\phi : \tilde{X} \to Y$ is of degree d, the natural projection map $pr_i : Z \to \tilde{X}$ is dominant and generically finite for any i. Hence, we may choose an irreducible component of Z such that the projection maps are dominant and generically finite. Denote X' a desingularisation of such an irreducible component of Z. Let $p_i : X' \to \tilde{X}$ be the natural projection maps and let $f : X' \to Y$ be the structural map of the fibered product. Let $I \in CH(X' \times \tilde{X})$ be the incidence cycle such that $I_* : CH_0(X') \to CH_0(\tilde{X})$ is given by $x \mapsto \sum_{i=1}^d p_i(x)$ for general $x \in X'$. Then, $I_* = \phi^* f_* : CH_0(X') \to CH_0(\tilde{X})$. By Mumford-type theorems [Voi14, 1.1.3], $I^* = f^* \phi_* : H^0(\tilde{X}, \Omega^l_{\tilde{X}}) \to H^0(X', \Omega^l_{X'})$ for $l > 0$. Since $H^{l,0}(Y) = 0$ for $l > 0$, and since I^* factorizes through $H^{l,0}(Y)$, we have $I^* = 0 : H^0(\tilde{X}, \Omega^l_{\tilde{X}}) \to H^0(X', \Omega^l_{X'})$. Hence, for any $l > 0$, we have

$$\sum_{i=1}^{d} p_i^* \sigma^l_{\tilde{X}} = 0$$

By the theory of symmetric polynomials, any symmetric polynomial without constant terms in $p_1^*\sigma_{\tilde{X}}, \ldots, p_d^*\sigma_{\tilde{X}}$ is zero. Hence,

$$\prod_{i=1}^{d}(T - p_i^*\sigma_{\tilde{X}}) = T^d.$$

Taking $T = p_1^*\sigma_{\tilde{X}}$, one gets $p_1^*\sigma_{\tilde{X}}^d = 0$. This implies in particular that the rank of σ_X is less than d. Hence, $d > n$. □

For the fibering genus of hyper-Kähler manifolds, Voisin proves the following result in [Voi21].

Theorem 3.1.6 (Voisin [Voi21])**.** *Let X be a Mumford-Tate very general hyper-Kähler manifold of dimension $2n$ with $n \geq 3$ and $b_{2,\mathrm{tr}}(X) \geq 5$. Then*

$$\mathrm{Fibgen}(X) \geq \min\{n+2, 2^{\lfloor \frac{b_{2,\mathrm{tr}}(X)-3}{2} \rfloor}\}.$$

The reader is referred to Section 1.4 and Section 1.4.5 for details about Mumford-Tate groups. In this chapter, we will present a sharper bound.

Theorem 3.1.7. *Let X be a Mumford-Tate very general hyper-Kähler manifold of dimension $2n$ and assume $b_{2,\mathrm{tr}}(X) \geq 5$. Then*

$$\mathrm{Fibgen}(X) \geq \min\left\{n + \left\lceil \frac{-1+\sqrt{8n-7}}{2} \right\rceil, 2^{\lfloor \frac{b_{2,\mathrm{tr}}(X)-3}{2} \rfloor}\right\}.$$

3.1.3 Measures of irrationality of K3 surfaces

We first recollect some known results on the assymptotic behaviors of the measures of irrationality of K3 surfaces.

Degree of irrationality

In [BDELU17], the following conjecture on the degree of irrationality of K3 surfaces is proposed.

Conjecture 3.1.8 ([BDELU17])**.** *Let $\{(S_d, L_d)\}_{d\in\mathbb{N}}$ be very general polarized K3 surfaces such that $L_d^2 = 2d-2$. Then*

$$\limsup_{d\to\infty} \mathrm{Irr}(S_d) = +\infty.$$

In the litterature, the d appearing in Conjecture 3.1.8 is called the *genus* of the K3 surface. To justify the name, for any smooth curve $C \subset S$ in the linear sytem $|L_d|$, it follows from tne adjunction formula that the arithemetic genus of C is d.

Remark 3.1.9. We can calculate the degree of irrationality of S_d when the genus d is small.

- When $d = 2$, the very general K3 surface S_2 is a double cover of $\mathbb{P}^2$ ramified along a sextic curve. Since K3 surfaces are not rational, the degree of irrationality of S_2 is 2.
- When $d = 3$, the very general K3 surface S_3 is a quartic surface in $\mathbb{P}^3$ and is not birational to a double cover of $\mathbb{P}^2$ since S_3 does not admit a rational involution. Projecting $S_3 \subset \mathbb{P}^3$ via a general point on S_3 gives a rational map of degree 3 of S_3 to $\mathbb{P}^2$. Hence, the degree of irrationality of S_3 is 3.

Despite the above examples, it is in general quite hard to determine the degree of irrationality of K3 surfaces.

One general result we know in the spirit of this conjecture is the following theorem proved in Stapleton's thesis [Stap17].

Theorem 3.1.10 (Stapleton [Stap17])**.** *There exists a constant $C > 0$ such that* $\mathrm{Irr}(S_d) \leq C\sqrt{d}$.

Fibering genus

The fibering genus of surfaces has already been studied by Ein and Lazarsfeld in [EL20], under the name *Konno invariant*. They give a good estimate of the asymptotic behavior of fibering genus of K3 surfaces as the genus tends to infinity.

Theorem 3.1.11 (Ein–Lazarsfeld [EL20])**.** *Let S_d be a polarized K3 surface of Picard rank* 1 *and genus d. Then*

$$\mathrm{Fibgen}(S_d) = \Theta(\sqrt{d}).$$

More precisely,

$$\sqrt{\frac{d}{2}} \leq \mathrm{Fibgen}(S_d) \leq 2\sqrt{2d}.$$

Corollary 3.1.12. *We have*

$$\lim_{d\to\infty} \mathrm{Fibgen}(S_d) = +\infty.$$

Fibering gonality

We do not know much about the fibering gonality of K3 surfaces. In this chapter, we will prove the following relation between the degree of irrationality and the fibering gonality of K3 surfaces.

Theorem 3.1.13. *Let $\{S_d\}_{d\in\mathbb{N}}$ be projective K3 surfaces such that the Picard group of S_d is generated by a line bundle with self intersection number $2d-2$. Then*

$$\limsup_{d\to\infty} \mathrm{Irr}(S_d) = +\infty \iff \limsup_{d\to\infty} \mathrm{Fibgon}(S_d) = +\infty.$$

The proof of Theorem 3.1.13 relies on Theorem 3.1.11 and the following theorem.

Theorem 3.1.14. *Let S be a projective K3 surface whose Picard number is* 1*. Then one of the following two cases holds:*
(a) $\mathrm{Irr}(S) = \mathrm{Fibgon}(S)$*;*
(b) $\mathrm{Fibgen}(S)^2 \leq \mathrm{Fibgon}(S)^{21}$.

Proof of Theorem 3.1.13 assuming Theorem 3.1.14. The implication $\Leftarrow$ is clear since $\mathrm{Fibgon}(S_d) \leq \mathrm{Irr}(S_d)$ for any d. Now let us prove the implication $\Rightarrow$. Theorem 3.1.11 ([EL20, Theorem B]) shows that $\lim_d \mathrm{Fibgen}(S_d) = +\infty$. If $\limsup_d \mathrm{Fibgon}(S_d) \neq +\infty$, then there exist constants C and D such that for every $d > D$ we have $\mathrm{Fibgon}(S_d)^{21/2} < C < \mathrm{Fibgen}(S_d)$. Hence for $d > D$, we must have $\mathrm{Irr}(S_d) = \mathrm{Fibgon}(S_d) < C^{2/21}$ by Theorem 3.1.14, which contradicts the assumption that $\limsup_d \mathrm{Irr}(S_d) = +\infty$. □

Conjecture 3.1.8 predicts that $\limsup_d \mathrm{Irr}(S_d) = +\infty$, so it should be expected that $\limsup_d \mathrm{Fibgon}(S_d) = +\infty$ as well.

3.1.4 Some lattice theoretic results about hyper-Kähler manifolds

In the course of the proof of Theorem 3.1.14, we obtain the following inequalities about the discriminant of Picard lattices of projective hyper-Kähler manifolds. Although we only use this proposition in the case of $K3$ surfaces and we only use one side of the inequalities, the following general form is of independent interest. Here $\mathrm{disc}(\mathrm{Pic}(X))$ denotes the discriminant of the Picard lattice of X with respect to the Beauville-Bogolov-Fujiki form and similarly for $\mathrm{disc}(\mathrm{Pic}(X'))$.

Proposition 3.1.15. *Let X and X' be deformation equivalent projective hyper-Kähler manifolds of dimension $2n$. Let $\phi: X \dashrightarrow X'$ be a dominant rational map. Then*

$$(\deg\phi)^{(\frac{1}{n}-2)b_{2,\mathrm{tr}}(X)} \leq \left|\frac{\mathrm{disc}(\mathrm{Pic}(X))}{\mathrm{disc}(\mathrm{Pic}(X'))}\right| \leq (\deg\phi)^{\frac{b_{2,\mathrm{tr}}(X)}{n}}.$$

If in the previous proposition X and X' are $K3$ surfaces, we can get a slightly better lower bound.

Proposition 3.1.16. *Let S and S' be projective $K3$ surfaces and let $\lambda(S) = \min\{\rho(S), b_{2,\mathrm{tr}}(S)\}$ where $\rho(S)$ is the Picard number of S. Let $\phi: S \dashrightarrow S'$ be a dominant rational map. Then*

$$(\deg\phi)^{-\lambda(S)} \leq \left|\frac{\mathrm{disc}(\mathrm{Pic}(S))}{\mathrm{disc}(\mathrm{Pic}(S'))}\right| \leq (\deg\phi)^{b_{2,\mathrm{tr}}(S)}.$$

3.2 Fibering genus of very general hyper-Kähler manifolds

Let X be a projective hyper-Kähler manifold of dimension $2n$. Let $f: X \dashrightarrow B$ be a fibration into curves and let $\tau: \tilde{X} \to X$ and $\tilde{f}: \tilde{X} \to B$ be a resolution of indeterminacy points. Let $\tilde{X}_b$ be a smooth fiber of $\tilde{f}$ over a general point $b \in B$. The exact sequence of vector bundles on $\tilde{X}_b$

$$0 \to N^*_{\tilde{X}_b/\tilde{X}} \to \Omega_{\tilde{X}|\tilde{X}_b} \to \Omega_{\tilde{X}_b} \to 0$$

induces an exact sequence

$$0 \to N^*_{\tilde{X}_b/\tilde{X}} \wedge \Omega_{\tilde{X}|\tilde{X}_b} \to \Omega^2_{\tilde{X}|\tilde{X}_b} \to \Omega^2_{\tilde{X}_b} \to 0.$$

Since $\tilde{X}_b$ is a curve, we have $\Omega^2_{\tilde{X}_b} = 0$. Therefore, $\Omega^2_{\tilde{X}|\tilde{X}_b} \cong N^*_{\tilde{X}_b/\tilde{X}} \wedge \Omega_{\tilde{X}|\tilde{X}_b}$ as vector bundles on $\tilde{X}_b$. Let us view $N^*_{\tilde{X}_b/\tilde{X}} \wedge \Omega_{\tilde{X}|\tilde{X}_b}$ as a subbundle of $N^*_{\tilde{X}_b/\tilde{X}} \otimes \Omega_{\tilde{X}|\tilde{X}_b}$. Then $(\tau^*(\sigma_X))|_{\tilde{X}_b} \in H^0(\tilde{X}_b, \Omega^2_{\tilde{X}|\tilde{X}_b})$ can be viewed as an element in $H^0(\tilde{X}_b, N^*_{\tilde{X}_b/\tilde{X}} \wedge \Omega_{\tilde{X}|\tilde{X}_b}) = \mathrm{Hom}_{\tilde{X}_b}(N_{\tilde{X}_b/\tilde{X}}, \Omega_{\tilde{X}|\tilde{X}_b})$. Therefore, $(\tau^*(\sigma_X))|_{\tilde{X}_b}$ induces a morphism $\phi_b: H^0(\tilde{X}_b, N_{\tilde{X}_b/\tilde{X}}) \to H^0(\tilde{X}_b, \Omega_{\tilde{X}_b})$. Therefore, we have a morphism

$$\sigma_b: T_{B,b} \to H^0(\tilde{X}_b, \Omega_{\tilde{X}_b}) = H^0(\tilde{X}_b, K_{\tilde{X}_b})$$

defined as the composition of the natural morphisms $T_{B,b} \to H^0(\tilde{X}_b, N_{\tilde{X}_b/\tilde{X}})$ and $\phi_b: H^0(\tilde{X}_b, N_{\tilde{X}_b/\tilde{X}}) \to H^0(\tilde{X}_b, \Omega_{\tilde{X}_b})$. Here, $H^0(\tilde{X}_b, K_{\tilde{X}_b}) \cong \mathbb{C}^g$ where $g = g(\tilde{X}_b)$ is the genus of the curve $\tilde{X}_b$. Let $\rho: T_{B,b} \to H^1(\tilde{X}_b, T_{\tilde{X}_b})$ be the Kodaira-Spencer map. In [Voi21], Voisin proves the following

Proposition 3.2.1 (Voisin [Voi21]). *(i) If $n \geq 2$, then $\tilde{X}_b$ is* not *hyperelliptic.*
(ii) The rank of σ_b is $\geq n$.
(iii) Let $I_b \subset H^0(\tilde{X}_b, K_{\tilde{X}_b}^{\otimes 2})$ be image of $\mathrm{Im}\sigma_b \otimes H^0(\tilde{X}_b, K_{\tilde{X}_b})$ under the multiplication map

$$\mu\colon H^0(\tilde{X}_b, K_{\tilde{X}_b}) \otimes H^0(\tilde{X}_b, K_{\tilde{X}_b}) \to H^0(\tilde{X}_b, K_{\tilde{X}_b}^{\otimes 2}).$$

Then $\rho(\ker\sigma_b) \subset H^1(\tilde{X}_b, T_{\tilde{X}_b})$ is orthogonal to I_b via the Serre pairing for $T_{\tilde{X}_b}$:

$$H^1(\tilde{X}_b, T_{\tilde{X}_b}) \otimes H^0(\tilde{X}_b, K_{\tilde{X}_b}^{\otimes 2}) \to H^1(\tilde{X}_b, K_{\tilde{X}_b}) \cong H^0(\tilde{X}_b, \mathscr{O}_{\tilde{X}_b}) \cong \mathbb{C}.$$

(iv) Assume that that $b_{2,\mathrm{tr}}(X) \geq 5$ and that the Mumford-Tate group of $H^2(X,\mathbb{Q})_{tr}$ is maximal. If $\rho\colon T_{B,b} \to H^1(\tilde{X}_b, T_{\tilde{X}_b})$ is not *injective, then $g \geq 2^{\lfloor \frac{b_{2,\mathrm{tr}}(X)-3}{2} \rfloor}$.*

The proofs of (i), (ii) and (iii) are explicitly written in [Voi21]. Although (iv) is not stated in [Voi21] with this generality, it is essentially proved there (see the proof of Lemma 2.13 and Lemma 2.15 in *loc. cit.*).

3.2.1 Proof of Theorem 3.1.7

For $n = 1$, the inequality is obvious, since $K3$ surfaces cannot be fibered into rational curves. From now on, let us assume $n \geq 2$ to be able to assume $\tilde{X}_b$ is not hyperelliptic (see Proposition 3.2.1 (i)). Let k be the corank of σ_b, i.e., $\mathrm{rank}\sigma_b = g - k$. We now prove

Lemma 3.2.2. *With notation as above, we have the following inequality*

$$\dim\ker\rho \geq 2n - 1 - (g-k) - \frac{k(k+1)}{2}.$$

Proof. We use the notation as in Proposition 3.2.1 (iii). By Serre duality, the orthogonality result in Proposition 3.2.1 (iii) implies that $\dim\rho(\ker\sigma_b) + \dim I_b \leq \dim H^0(\tilde{X}_b, K_{\tilde{X}_b}^{\otimes 2})$, that is,

$$\mathrm{codim}(I_b \subset H^0(\tilde{X}_b, K_{\tilde{X}_b}^{\otimes 2})) \leq \dim\rho(\ker\sigma_b). \tag{3.1}$$

The multiplication map $\mu\colon H^0(\tilde{X}_b, K_{\tilde{X}_b}) \otimes H^0(\tilde{X}_b, K_{\tilde{X}_b}) \to H^0(\tilde{X}_b, K_{\tilde{X}_b}^{\otimes 2})$ factors through the symmetric product $\mathrm{Sym}^2 H^0(\tilde{X}_b, K_{\tilde{X}_b})$. Let $\mathrm{Im}\,\sigma_b \cdot H^0(\tilde{X}_b, K_{\tilde{X}_b})$ denote the image of $\mathrm{Im}\,\sigma_b \otimes H^0(\tilde{X}_b, K_{\tilde{X}_b})$ under the canonical symmetrization map $\mathrm{pr}\colon H^0(\tilde{X}_b, K_{\tilde{X}_b}) \otimes H^0(\tilde{X}_b, K_{\tilde{X}_b}) \to \mathrm{Sym}^2 H^0(\tilde{X}_b, K_{\tilde{X}_b})$. Recall that k is the corank of σ_b, that is, $\mathrm{codim}(\mathrm{Im}\,\sigma_b \subset H^0(\tilde{X}_b, K_{\tilde{X}_b})) = k$. Then we have

$$\mathrm{codim}(\mathrm{Im}\,\sigma_b \cdot H^0(\tilde{X}_b, K_{\tilde{X}_b})) \subset \mathrm{Sym}^2 H^0(\tilde{X}_b, K_{\tilde{X}_b})) = \frac{k(k+1)}{2}. \tag{3.2}$$

On the other hand, by Max Noether theorem (see [ACGH85, Chapter III, §2] or [Voi03]), the multiplication map $\mu'\colon \mathrm{Sym}^2 H^0(\tilde{X}_b, K_{\tilde{X}_b}) \to H^0(\tilde{X}_b, K_{\tilde{X}_b}^{\otimes 2})$ is surjective, since $\tilde{X}_b$ is not hyperelliptic. Taking into account the fact that I_b is the image of $\mathrm{Im}\,\sigma_b \cdot H^0(\tilde{X}_b, K_{\tilde{X}_b})$ under the map μ', we have the following inequality

$$\mathrm{codim}(I_b \subset H^0(\tilde{X}_b, K_{\tilde{X}_b}^{\otimes 2})) \leq \mathrm{codim}(\mathrm{Im}\,\sigma_b \cdot H^0(\tilde{X}_b, K_{\tilde{X}_b}) \subset \mathrm{Sym}^2 H^0(\tilde{X}_b, K_{\tilde{X}_b})). \tag{3.3}$$

Combining (3.1), (3.2) and (3.3), we get $\dim\rho(\ker\sigma_b) \leq \frac{k(k+1)}{2}$, from which we deduce that

$$\dim\ker\rho \geq \dim\ker\sigma_b - \frac{k(k+1)}{2} = 2n - 1 - (g-k) - \frac{k(k+1)}{2}.$$

□

Proof of Theorem 3.1.7. Assuming that $g(\tilde{X}_b) < 2^{\lfloor \frac{b_{2,\mathrm{tr}}(X)-3}{2} \rfloor}$, we have to prove $g \geq n + \lceil \frac{-1+\sqrt{8n-7}}{2} \rceil$. By Proposition 3.2.1 (ii), (iv) and Lemma 3.2.2, we have the following constraints on g and k:

$$\begin{cases} k \geq 0 \\ g-k-n \geq 0 \\ 2n-1-(g-k)-\frac{k(k+1)}{2} \leq 0. \end{cases}$$

In order to find the minimal possible value of g under these constraints, we make the following discussion according to the values of k.

- When $0 \leq k \leq \frac{-1+\sqrt{8n-7}}{2}$, we have $k - \frac{k(k+1)}{2} + 2n - 1 \geq n + k$. Hence, the minimal possible value of g in this domain is the minimum of $k - \frac{k(k+1)}{2} + 2n - 1$ with $0 \leq k \leq \frac{-1+\sqrt{8n-7}}{2}$, which is $n + \frac{-1+\sqrt{8n-7}}{2}$.
- When $k \geq \frac{-1+\sqrt{8n-7}}{2}$, we have $k - \frac{k(k+1)}{2} + 2n - 1 \leq n + k$. Hence, the minimal possible value of g in this domain is the minimum of $n + k$ with $k \geq \frac{-1+\sqrt{8n-7}}{2}$, which is $n + \frac{-1+\sqrt{8n-7}}{2}$.

Since g and k are integers, we find $g \geq n + \lceil \frac{-1+\sqrt{8n-7}}{2} \rceil$, as desired.

□

Remark 3.2.3. Our proof relies on the inequality (3.3) which only uses the surjectivity of the multiplication map μ'. With more information on the geometry of the canonical embedding, and in particular, on the gonality of the fibers, we could get a better estimate in Theorem 3.1.7.

3.3 Relations between birational invariants of $K3$ surfaces

In this section, we are going to prove Theorem 3.1.14 that relates the three birational invariants, namely, the degree of irrationality, the fibering gonality and the fibering genus, of projective $K3$ surfaces of Picard number 1.

3.3.1 A factorization

Let S be a smooth projective surface and let $f\colon S \dashrightarrow B$ be a fibration into curves over a smooth base B. After a resolution of inderminacies of f and replacing S by another birational model, we may assume $f\colon S \to B$ is a morphism. Let d be the gonality of the general fiber of f, so the general fiber C admits a degree d morphism from C to $\mathbb{P}^1$. A standard argument shows that we can spread this morphism into a family up to a generically finite base change.

Lemma 3.3.1. *There is a generically finite morphism $\pi\colon B' \to B$ and a degree d dominant rational map $\psi\colon S \times_B B' \dashrightarrow \mathbb{P}^1 \times B'$ over B'.*

Proof. Let B_0 be the smooth locus of $f\colon S \to B$ and let $f_0\colon S_0 \to B_0$ be the restriction of f on the smooth locus. Let $p\colon \mathrm{Pic}^d(S_0/B_0) \to B_0$ be the degree d relative Picard variety of

$f_0 \colon S_0 \to B_0$. By the assumption on the general fiber of $f \colon S \to B$, the restriction of the map $p \colon \mathrm{Pic}^d(S_0/B_0) \to B_0$ to the Brill-Noether locus in $\mathrm{Pic}^d(S_0/B_0)$ of the linear systems of degree d and dimension 1 is dominant. Let B_0' be a general reduced irreducible subscheme of $\mathrm{Pic}^d(S_0/B_0)$ that is dominant and generically finite over B_0 by p. Let us take B' to be a completion of B_0'. Then by construction, the universal line bundle restricted to $S_0 \times_B B_0'$ gives a dominant rational map $\psi \colon S \times_B B' \dashrightarrow \mathbb{P}^1_{B'}$ of degree d, as desired. □

It is natural to ask if $\psi \colon S \times_B B' \dashrightarrow \mathbb{P}^1 \times B'$ over B' descends to a rational map $\psi_B \colon S \dashrightarrow \mathbb{P}^1 \times B$ over B. A moment of thinking will convince us that we are asking too much, because $\pi \colon B' \to B$ is in general not a Galois cover. We make the following construction. Let n be the degree of the morphism $\pi \colon B' \to B$. Consider the n-th self fibred product of B' over B: $B' \times_B \cdots \times_B B'$. define B'' to be the closure in $B' \times_B \cdots \times_B B'$ of the set

$$\{(x_1, \ldots, x_n) \in B' \times_B \cdots \times_B B' \colon x_1, \ldots, x_n \text{ are distinct } \}.$$

Then $\pi' \colon B'' \to B$ is of degree $n!$ and the symmetric group $\mathfrak{S}_n$ permuting the components of $B' \times_B \cdots \times_B B'$ acts on an open dense subset of B''. The rational map $\psi \colon S \times_B B' \dashrightarrow \mathbb{P}^1 \times B'$ over B' given in Lemma 3.3.1 can be extended to a rational map

$$\psi' \colon S \times_B B'' \dashrightarrow (\mathbb{P}^1)^n \times B'' \tag{3.4}$$

over B'' in a natural way: let $x \in S$ and let $y = (y_1, \ldots, y_n) \in B''$ be general points, we define $\psi'(x,y) = (\psi(x,y_1), \ldots, \psi(x,y_n), y)$. Moreover, the symmetric group $\mathfrak{S}_n$ acts canonically on both sides of (3.4) in the following way. To define the action of $\mathfrak{S}_n$ on $S \times_B B''$, we let $\mathfrak{S}_n$ act trivially on S and act as permutations of components of B''; and to define the action on $(\mathbb{P}^1)^n \times B''$, we let $\mathfrak{S}_n$ act as permutations of components for both $(\mathbb{P}^1)^n$ and B''. It is clear from the construction that $\psi' \colon S \times_B B'' \dashrightarrow (\mathbb{P}^1)^n \times B''$ is $\mathfrak{S}_n$-equivariant. Thus ψ' induces a rational map $S \dashrightarrow ((\mathbb{P}^1)^n \times B'')/\mathfrak{S}_n$ over B. Let S' be the image of this map. Thus we get a dominant rational map

$$\phi \colon S \dashrightarrow S'.$$

Proposition 3.3.2. *S' is a surface and the degree of ϕ divides d. Furthermore, if $n \geq 2$, the general fiber of $S' \dashrightarrow B$ is of geometric genus $\leq (\frac{d}{\deg \phi} - 1)^2$.*

Proof. Over the general point $b = (b_1, \ldots, b_n) \in B''$, ψ' is given by the morphism $\psi'_b \colon C \to (\mathbb{P}^1)^n$ induced by the n morphisms $C \to \mathbb{P}^1$ of degree d corresponding to the points $b_i \in B'$, where C is the fiber of $f \colon S \dashrightarrow B$ oveer $\pi'(b) \in B$. Let C' be the image of ψ'_b. Then the fiber of $S' \dashrightarrow B$ over $\pi'(b) \in B$ is C' by construction. Thus S' is a surface, and the degree of ϕ is the degree of C over C', which divides d. This proves the first statement. To prove the second, we need to prove the geometric genus of C' is $\leq (\frac{d}{\deg \phi} - 1)^2$. Since C' is a curve of multi-degree $(\frac{d}{\deg \phi}, \ldots, \frac{d}{\deg \phi})$ in $(\mathbb{P}^1)^n$, we can use Lemma 3.3.3 below concerning algebraic curves in $(\mathbb{P}^1)^n$. □

Lemma 3.3.3. *Let $n \geq 2$. Let C be an integral curve in $(\mathbb{P}^1)^n$ of multi-degree $(d, \ldots, d)$. Then the geometric genus of C is less than or equal to $(d-1)^2$.*

Proof. We prove it by induction on n. When $n = 2$, it is the adjunction formula. Now assume that any curve $C' \subset (\mathbb{P}^1)^{n-1}$ of multi-degree $(e, \ldots, e)$ has geometric genus $\leq (e-1)^2$. Consider the projection $C \to C'' \subset (\mathbb{P}^1)^{n-1}$ to the first $n-1$ components. The degree of $C \to C''$ is of the form d/e, for some e. Hence, $C'' \subset (\mathbb{P}^1)^{n-1}$ is a curve of multi-degree

$(e,\dots,e)$, hence it has geometric genus $\leq (e-1)^2$ by induction assumption. Let $\tilde{C}$ and $\tilde{C}''$ be the normalization of C and C'' respectively. Then $\tilde{C}$ is birational to its image C''' in $\tilde{C}''\times\mathbb{P}^1$, where the map to the second component is given by the composition map $\tilde{C}\to C\xrightarrow{i_n}\mathbb{P}^1$. Here, the map $i_n\colon C\to\mathbb{P}^1$ is the projection map to the n-th component. Note that C''' is of multi-degree $(d/e,d)$ in $\tilde{C}''\times\mathbb{P}^1$ and note that $NS(C''\times\mathbb{P}^1)=NS(C'')\oplus NS(\mathbb{P}^1)$. Adjunction formula and $g(C'')\leq(e-1)^2$ give us $p_a(C''')=\frac{d(g(C'')+d-1)}{e}-d+1\leq d(e+d/e-2)-d+1\leq(d-1)^2$ since $1\leq e\leq d$. Hence, $p_g(C)=g(\tilde{C})=p_g(C''')\leq p_a(C''')\leq(d-1)^2$, as desired. □

3.3.2 Rational maps between $K3$ surfaces

We treat rational maps between $K3$ surfaces in this part. Let S (resp. S') be a projective $K3$ surface whose Picard group is generated by an ample line bundle of degree $2D-2$ (resp. $2D'-2$). Let $\phi\colon S\dashrightarrow S'$ be a dominant rational map.

Proposition 3.3.4. *We have the following inequality*

$$\frac{1}{(\deg\phi)^{21}}\leq\frac{D-1}{D'-1}\leq(\deg\phi)^{21}.$$

Proof. Let $\tau\colon\tilde{S}\to S$ and $\tilde{\phi}\colon\tilde{S}\to S'$ be a resolution of indeterminacy points of $\phi\colon S\dashrightarrow S'$. Let T (resp. T') be the lattice $H^2(S,\mathbb{Z})_{tr}$ (resp. $H^2(S',\mathbb{Z})_{tr}$) endowed with the intersection form. For a positive integer e, define $T'(e)$ to be the lattice T' with the quadratic form multiplied by e. For example, with this notation, the sublattice eT' of T' is isometric to $T'(e^2)$ as lattices. The image E of the morphism $\tilde{\phi}^*\colon H^2(S',\mathbb{Z})_{tr}\to H^2(\tilde{S},\mathbb{Z})_{tr}\cong T$, viewed as a sublattice of T, is isometric to $T'(\deg\phi)$. The isomorphism $H^2(\tilde{S},\mathbb{Z})_{tr}\cong T$ is because $\tau^*\colon H^2(S,\mathbb{Z})_{tr}\to H^2(\tilde{S},\mathbb{Z})_{tr}$ is an isomorphism. We thus get the following equalities

$$[T\colon E]^2=\left|\frac{\mathrm{disc}(E)}{\mathrm{disc}(T)}\right|=(\deg\phi)^{21}\left|\frac{\mathrm{disc}(T')}{\mathrm{disc}(T)}\right|=(\deg\phi)^{21}\frac{D'-1}{D-1}. \tag{3.5}$$

Here, we have used the fact that the lattice $H^2(S,\mathbb{Z})$ is unimodular since S is a surface, and the fact that the orthogonal complement of a primitive sublattice in a unimodular lattice and the sublattice itself have the same discriminant, up to sign ([Nik80, Proposition 1.6.1]). On the other hand, we have

Lemma 3.3.5. *The morphism of abelian groups $\tilde{\phi}_*\colon H^2(\tilde{S},\mathbb{Z})_{tr}\to H^2(S',\mathbb{Z})_{tr}$ is injective and sends E onto $(\deg\phi)T'$.*

Proof. The projection formula shows that $\tilde{\phi}_*\tilde{\phi}^*=\deg\phi\cdot Id$. Hence, $\tilde{\phi}_*$ sends E onto $(\deg\phi)T'$. By Lemma 3.3.7 and the fact that $\tilde{\phi}_*$ is surjective with $\mathbb{Q}$-coefficients, the kernel of $\tilde{\phi}_*$ is of torsion. But $H^2(\tilde{S},\mathbb{Z})_{tr}$ is torsion-free, as $\tilde{S}$ is simply connected. We conclude that the kernel of $\tilde{\phi}_*$ is zero, as desired. □

Now Lemma 3.3.5 implies that

$$1\leq[T\colon E]\leq[T'\colon(\deg\phi)T']=(\deg\phi)^{21}. \tag{3.6}$$

Proposition 3.3.4 follows by combining (3.5) and (3.6). □

Remark 3.3.6. One can similarly prove the more general result on hyper-Kähler manifolds, namely Proposition 3.1.15. The detailed proof is given in Section 3.4. A sharper lower bound will also be given there.

3.3.3 Proof of Theorem 3.1.14;

Let $f\colon S \dashrightarrow B = \mathbb{P}^1$ be a fibration into curves realizing the fibering gonality of S. After a resolution of indeterminacies of f, we get a dominant morphism $\tilde{f}\colon \tilde{S} \to B$ whose general fiber is of gonality $d = \mathrm{Fibgon}(S)$. In Section 3.3.1, we constructed a surface S' that is a fibration over B into curves and a dominant rational map $\phi\colon \tilde{S} \dashrightarrow S'$ over B of degree dividing d. The Kodaira dimension $\kappa(S')$, the irregularity $q(S')$ and the geometric genus $p_g(S')$ of S' cannot exceed those of S since S' is dominated by S. By Enriques-Kodaira classification of algebraic surfaces [Enr49, Kod64-68], S' can only be birational to $\mathbb{P}^2$, an Enriques surface or a K3 surface.

If S' is a rational surface, then $\mathrm{Irr}(S) \leq \deg\phi \leq d = \mathrm{Fibgon}(S)$. Here, the inequality $\deg\phi \leq d$ is because of Proposition 3.3.2. But clearly $\mathrm{Fibgon}(S) \leq \mathrm{Irr}(S)$. We get the equality. This is case (a) of the theorem.

If S' is birational to an Enriques surface S''. After a birational modification of $\tilde{S}$, there is a dominant morphism $g\colon \tilde{S} \to S''$. Since $\tilde{S}$ is simply-connected, g factors through the universal covering S''' of S''. S''' is the K3 cover of the Enriques surface S''. The Picard number of S''' is at least 10 since the Picard number of the Enriques surface S'' is 10.

Lemma 3.3.7. *Let $\phi\colon X \dashrightarrow X'$ be a dominant rational map between projective hyper-Kähler manifolds of the same dimension. Then $b_{2,\mathrm{tr}}(X) = b_{2,\mathrm{tr}}(X')$.*

Proof. Let $\tau\colon \tilde{X} \to X$ and $\tilde{\phi}\colon \tilde{X} \to X'$ be a resolution of indeterminacy points of ϕ. Then $\tau^*\colon H^2(X,\mathbb{Q})_{tr} \to H^2(\tilde{X},\mathbb{Q})_{tr}$ is an isomorphism and $\tilde{\phi}^*\colon H^2(X',\mathbb{Q})_{tr} \to H^2(\tilde{X},\mathbb{Q})_{tr}$ is injective. They are moreover both morphisms of Hodge structures. But $H^2(X,\mathbb{Q})_{tr}$ and $H^2(X',\mathbb{Q})_{tr}$ are simple Hodge structures. This implies that $\tilde{\phi}^*\colon H^2(X',\mathbb{Q})_{tr} \to H^2(\tilde{X},\mathbb{Q})_{tr}$ is an isomorphism and hence the result. □

Lemma 3.3.7 shows that the Picard number of S''' can only be 1 since it is dominated by S. This gives us a contradiction. The case where S' is birational to an Enriques surface is thus excluded.

In the rest of this section, we discuss the case when S' is birational to a $K3$ surface. By changing the birational model, we may assume S' *is* a K3 surface. By Lemma 3.3.7, the Picard number of S' is also 1. The following proposition shows that in our situation the Case (b) of Theorem 3.1.14 holds, which concludes the proof of Theorem 3.1.14.

Proposition 3.3.8. *The following inequality holds:*

$$\mathrm{Fibgen}(S) \leq \mathrm{Fibgon}(S)^{21/2}.$$

Proof. Let D and D' be the degrees of the $K3$ surfaces S and S', respectively. Let C' be the general fiber of $S' \dashrightarrow B$ as in Section 3.3.1. Then we have the following inequalities

$$\begin{aligned}
\left(\frac{\mathrm{Fibgon}(S)}{\deg\phi} - 1\right)^2 &\geq p_g(C') && \text{by Proposition 3.3.2}\\
&\geq \sqrt{\frac{D'}{2}} && \text{by Ein-Lazarsfeld's theorem (Theorem 3.1.11)}\\
&\geq \sqrt{\frac{D}{2(\deg\phi)^{21}}} && \text{by Proposition 3.3.4}\\
&\geq \frac{\mathrm{Fibgen}(S)}{4(\deg\phi)^{21/2}} && \text{by Ein-Lazarsfeld's theorem (Theorem 3.1.11).}
\end{aligned}$$

Note that $\mathrm{Fibgon}(S) \geq 2\deg\phi$. Proposition 3.3.8 follows from these inequalities. □

3.4 Some inequalities about Picard lattices of hyper-Kähler manifolds

We prove in this section Propositions 3.1.15 and 3.1.16.

Proof of Proposition 3.1.15. Let $\tau\colon \tilde{X} \to X$ and $\tilde{\phi}\colon \tilde{X} \to X'$ be a resolution of indeterminacy points of $\phi\colon X \dashrightarrow X'$. Let T (resp. T') be the lattice $H^2(X,\mathbb{Z})_{tr}$ (resp. $H^2(X',\mathbb{Z})_{tr}$) endowed with the Beauville-Bogomolov-Fujiki form. As in the proof of Proposition 3.3.4, for a positive integer e, define $T'(e)$ to be the lattice T' with the quadratic form multiplied by e. We claim that the image E of the morphism $\tilde{\phi}^*\colon H^2(X',\mathbb{Z})_{tr} \to \tau^* H^2(X,\mathbb{Z})_{tr} \cong T$, viewed as a sublattice of T, is isometric to $T'((\deg\phi)^{\frac{1}{n}})$. This follows from the equalities $[q_X(\tilde{\phi}^*\alpha)]^n = c_X \cdot (\int_X \tilde{\phi}^*\alpha^{2n}) = (\deg\phi)\cdot c_X \cdot (\int_{X'} \alpha^{2n}) = (\deg\phi)\cdot [q_{X'}(\alpha)]^n$, where $c_X = c_{X'}$ is the Fujiki constant for the deformation class of X and we have viewed $\tilde{\phi}^*\alpha$ as an element in $H^2(X,\mathbb{Z})$ via the isomorphism $\tau_* : \tau^* H^2(X,\mathbb{Z})_{tr} \to H^2(X,\mathbb{Z})_{tr}$. Now the claim implies the following equalities

$$[T\colon E]^2 = \left|\frac{\mathrm{disc}(E)}{\mathrm{disc}(T)}\right| = (\deg\phi)^{\frac{b_{2,\mathrm{tr}}(X)}{n}} \cdot \left|\frac{\mathrm{disc}(T')}{\mathrm{disc}(T)}\right| = (\deg\phi)^{\frac{b_{2,\mathrm{tr}}(X)}{n}} \cdot \left|\frac{\mathrm{disc}(\mathrm{Pic}(X'))}{\mathrm{disc}(\mathrm{Pic}(X))}\right|. \quad (3.7)$$

On the other hand, with a similar argument to Lemma 3.3.5, we prove that the morphism of abelian groups $\tilde{\phi}_*\colon H^2(X,\mathbb{Z})_{tr} \to H^2(X',\mathbb{Z})_{tr}$ is injective and sends E to $(\deg\phi)T'$. Hence,

$$1 \le [T\colon E] \le [T'\colon (\deg\phi)T'] = (\deg\phi)^{b_{2,\mathrm{tr}}(X)}. \quad (3.8)$$

The proposition follows by combining (3.7) and (3.8). □

Proof of Proposition 3.1.16. The only thing that needs proving, in the view of Proposition 3.1.15, is the following inequality

$$\frac{1}{(\deg\phi)^{\rho(X)}} \le \left|\frac{\mathrm{disc}(\mathrm{Pic}(S))}{\mathrm{disc}(\mathrm{Pic}(S'))}\right|.$$

Let $\tau\colon \tilde{S} \to S$ and $\tilde{\phi}\colon \tilde{S} \to S'$ be a resolution of indeterminacy points of $\phi\colon S \dashrightarrow S'$. Via the morphism $\tilde{\phi}^*\colon \mathrm{Pic}(S') \to \mathrm{Pic}(\tilde{S})$, we can view $\tilde{\phi}^*\mathrm{Pic}(S')$ as a sublattice of $\mathrm{Pic}(\tilde{S})$. The sublattice $\tilde{\phi}^*\mathrm{Pic}(S')$ is isomorphic to $\mathrm{Pic}(S')(\deg\phi)$, since $\tilde{\phi}^*\alpha \cup \tilde{\phi}^*\beta = \tilde{\phi}^*(\alpha\cup\beta) = \deg\phi\cdot(\alpha\cup\beta)$ for $\alpha,\beta \in \mathrm{Pic}(S')$. Thus

$$\mathrm{disc}(\tilde{\phi}^*(\mathrm{Pic}(S'))) = (\deg\phi)^{\rho(S)}\mathrm{disc}(\mathrm{Pic}(S')). \quad (3.9)$$

Lemma 3.4.1. *The sublattice* $\ker(\tilde{\phi}_*\colon \mathrm{Pic}(\tilde{S}) \to \mathrm{Pic}(S'))$ *of* $\mathrm{Pic}(\tilde{S})$ *is the orthogonal complement of* $\tilde{\phi}^*(\mathrm{Pic}(S'))$ *in* $\mathrm{Pic}(\tilde{S})$.

Proof. Let $\alpha \in \mathrm{Pic}(\tilde{S})$. Let us show that $\tilde{\phi}_*\alpha = 0$ if and only if for any $\beta \in \mathrm{Pic}(S')$, we have $\alpha \cup \tilde{\phi}^*\beta = 0$ in $H^4(\tilde{S},\mathbb{Z})$. The projection formula gives $\tilde{\phi}_*(\alpha\cup\tilde{\phi}^*\beta) = (\tilde{\phi}_*\alpha)\cup\beta$ in $H^4(S',\mathbb{Z})$. Hence, if $\tilde{\phi}_*\alpha = 0$, then $\tilde{\phi}_*(\alpha\cup\tilde{\phi}^*\beta) = 0$. But $\tilde{\phi}_*\colon H^4(\tilde{S},\mathbb{Z}) \to H^4(S',\mathbb{Z})$ is an isomorphism, we must have $\alpha\cup\tilde{\phi}^*\beta = 0$. Conversely, if $\alpha\cup\tilde{\phi}^*\beta = 0$ for every $\beta \in \mathrm{Pic}(S')$, still by the projection formula, we get $(\tilde{\phi}_*\alpha)\cup\beta = 0$, which implies that $\tilde{\phi}_*\alpha = 0$ since the intersection product map is nondegenerate on $\mathrm{Pic}(S')$. □

Taking into account the fact that the intersection map on $H^2(\tilde{S},\mathbb{Z})$ is nondegenerate on $\tilde{\phi}^*\mathrm{Pic}(S')$, Lemma 3.4.1 implies that $\ker(\tilde{\phi}_*)\oplus\tilde{\phi}^*(\mathrm{Pic}(S'))$ is a direct sum and that $\ker(\tilde{\phi}_*)\oplus\tilde{\phi}^*(\mathrm{Pic}(S'))$ is of *finite* index in the abelian group $\mathrm{Pic}(\tilde{S})$. Hence,

$$\left|\frac{\mathrm{disc}(\ker(\tilde{\phi}_*))\cdot\mathrm{disc}(\tilde{\phi}^*(\mathrm{Pic}(S')))}{\mathrm{disc}(\mathrm{Pic}(\tilde{S}))}\right| = [\mathrm{Pic}(\tilde{S}):\ \ker(\tilde{\phi}_*)\oplus\tilde{\phi}^*(\mathrm{Pic}(S'))]^2. \tag{3.10}$$

Since $\tilde{\phi}_*\colon \mathrm{Pic}(\tilde{S})\to\mathrm{Pic}(S'))$ sends $\ker(\tilde{\phi}_*)\oplus\tilde{\phi}^*(\mathrm{Pic}(S'))$ onto $(\deg\phi)\mathrm{Pic}(S')$, and since the induced morphism

$$\overline{\tilde{\phi}_*}\colon \mathrm{Pic}(\tilde{S})/\ker\tilde{\phi}_*\to\mathrm{Pic}(S')$$

is injective, we have

$$\begin{aligned}[\mathrm{Pic}(\tilde{S}):(\ker(\tilde{\phi}_*)\oplus\tilde{\phi}^*(\mathrm{Pic}(S')))] &= [(\mathrm{Pic}(\tilde{S})/\ker\tilde{\phi}_*):\overline{\tilde{\phi}^*(\mathrm{Pic}(S'))}]\\ &\le [\mathrm{Pic}(S'):(\deg\phi)\mathrm{Pic}(S')]\\ &= (\deg\phi)^{\rho(S)},\end{aligned} \tag{3.11}$$

where $\overline{\tilde{\phi}^*(\mathrm{Pic}(S'))}$ is the image of $\tilde{\phi}^*(\mathrm{Pic}(S'))$ in $\mathrm{Pic}(\tilde{S})/\ker\tilde{\phi}_*$. Notice the following Lemma.

Lemma 3.4.2. $|\mathrm{disc}(\mathrm{Pic}(\tilde{S}))| = |\mathrm{disc}(\mathrm{Pic}(S))|$.

Proof. $\tilde{S}$ is obtained by a sequence of blowing-ups of points from S. Therefore,

$$\mathrm{Pic}(\tilde{S}) = \tau^*\mathrm{Pic}(S)\oplus\bigoplus_i\mathbb{Z}E_i,$$

where E_i is the *total transform* in $\tilde{S}$ of the exceptional divisor of the i-th blowing-up. We have the following formula for the intersection numbers of E_i:

$$E_i.E_j = \begin{cases} -1 & \text{if } i=j\\ 0 & \text{otherwise.}\end{cases}$$

Hence, $\mathrm{disc}(\mathrm{Pic}(\tilde{S})) = \mathrm{disc}(\mathrm{Pic}(S))\cdot\mathrm{disc}(\oplus_i\mathbb{Z}E_i) = \pm\mathrm{disc}(\mathrm{Pic}(S))$, as desired. □

The proposition now follows from Lemma 3.4.2 and from inequalities (3.9), (3.10) and (3.11), noticing that $\mathrm{disc}(\ker(\tilde{\phi}_*))\ge 1$. □

3.5 Related results

The study of rational maps between K3 surfaces is an intriguing question. One of the circulated expectations in the area is the following question.

Problem 3.5.1. *Let S be a very general projective K3 surface. Let $\phi : S \dashrightarrow S'$ be a dominant rational map to another K3 surface S'. Is that true that ϕ can only be an isomorphism (i.e. the degree of ϕ is 1)?*

As is clear from the proof of Theorem 3.1.14, if Problem 3.5.1 has a positive answer, then the Case (b) in Theorem 3.1.14 can be improved to be $\mathrm{Fibgon}(S)^2\ge\mathrm{Fibgen}(S)$. We do not expect a purely lattice theoretic answer.

We can also study self-rational maps of hyper-Kähler manifolds, the higher dimensional generalization of K3 surfaces. As we can see from the following theorem, self-rational maps of hyper-Kähler manifolds has very restrictive numeric properties.

Theorem 3.5.2. *Let X be a projective hyper-Kähler manifold of dimension $2n$ such that $b_{2,tr}(X)$ is odd. Let σ_X be its holomorphic symplectic form. Let $\phi : X \dashrightarrow X$ be a dominant self-rational map. Then the degree of ϕ is of the form k^{2n} where k is an integer. In this case, $\phi^*\sigma_X = \pm k\sigma_X$.*

Proof. Let $\tau : \tilde{X} \to X$ and $\tilde{\phi} : \tilde{X} \to X$ be a resolution of indeterminacies of $\phi : X \dashrightarrow X$. Let T be the lattice $H^2(X,\mathbb{Z})_{tr}$ endowed with the Beauville–Bogomolov–Fujiki form q_X. For a positive integer e, define $T(e)$ to be the lattice T with the quadratic form multiplied by e. We claim that the image E of the morphism $\tilde{\phi}^* : H^2(X,\mathbb{Z})_{tr} \to \tau^*H^2(X,\mathbb{Z})_{tr} \cong T$, viewed as a sublattice of T, is isometric to $T((\deg\phi)^{\frac{1}{n}})$. This follows from the equalities $[q_X(\tilde{\phi}^*\alpha)]^n = c_X \cdot (\int_X \tilde{\phi}^*\alpha^{2n}) = (\deg\phi)\cdot c_X \cdot (\int_X \alpha^{2n}) = (\deg\phi)\cdot[q_X(\alpha)]^n$, where c_X is the Fujiki constant for the deformation class of X and we have viewed $\tilde{\phi}^*\alpha$ as an element in $H^2(X,\mathbb{Z})$ via the isomorphism $\tau_* : \tau^*H^2(X,\mathbb{Z})_{tr} \to H^2(X,\mathbb{Z})_{tr}$.. The claim already implies that $\deg\phi$ is of the form m^n with m an integer. Let us show that m is a perfect square. Note that the claim also implies the following equalities

$$[T:E]^2 = \left|\frac{\mathrm{disc}(E)}{\mathrm{disc}(T)}\right| = (\deg\phi)^{\frac{b_{2,tr}(X)}{n}} \cdot \left|\frac{\mathrm{disc}(T)}{\mathrm{disc}(T)}\right| = (\deg\phi)^{\frac{b_{2,tr}(X)}{n}} = m^{b_{2,tr}(X)}. \tag{3.12}$$

Since $b_{2,tr}(X)$ is assumed to be odd, comparing the two sides of the equality shows that m is a perfect square. Hence, the degree of ϕ is of the form k^{2n}.

Therefore, for each $\alpha \in H^2(X,\mathbb{Z})_{tr}$, we have $q_X(\tilde{\phi}^*\alpha) = k^2 q_X(\alpha)$. Therefore, the morphism $\psi := \frac{1}{k}\phi^* : H^2(X,\mathbb{Q})_{tr} \to H^2(X,\mathbb{Q})_{tr}$ is an Hodge isometry. Since $b_{2,tr}(X)$ is assumed to be odd, ψ has an eigenvector v with ± 1 as its eigenvalue. But now the fact that $H^2(X,\mathbb{Q})_{tr}$ is a *simple* Hodge structure implies that $\psi = \pm Id$. Theorem 3.5.2 then follows. □

Example 3.5.3. (i) Assume that X admits a Lagrangian fibration $f : X \to B$ with a (rational) zero section. Let $x \in X$ be a general point of X and $b = f(x) \in B$. The fact that the general fiber of the fibration is Lagrangian implies that

$$\sigma_{X,x} = \alpha \wedge f^*\beta,$$

where $\alpha \in \Omega_{X_b,x}$ and $\beta \in \Omega_{B,b}$. Since the general fiber X_b is an abelian variety of dimension n, the self map $m_{k,b} : X_b \to X_b$ defined as the multiplication by a natural number k is of degree k^{2n}, and $m_{k,b}^*\alpha = k\alpha$. The existence the global zero section of this Lagrangian fibration implies that $m_{k,b} : X_b \to X_b$ can be defined globally as a self-rational map $m_k : X \dashrightarrow X$. The degree of m_k is k^{2n} and $m_k^*\sigma_X = k\sigma_X$.
(ii) Let X be the Fano variety of lines of a cubic fourfold Y. The following self-rational map $\phi : X \dashrightarrow X$, called the Voisin map, is constructed in [Voi04] as follows. Let $\ell \in X$ be a general line in Y. Then there is a unique plane in $\mathbb{P}^5$ that passes ℓ twice. The intersection of Y and this plane is of the form $2\ell + \ell'$ where ℓ' is another line in Y. Then $\phi(\ell)$ is defined to be ℓ'. As is shown in [Voi04], $\phi : X \dashrightarrow X$ is of degree $16 = 2^4$ and $\phi^*\sigma_X = -2\sigma_X$.

Chapter 4

On the Geometry of the Higer Dimension Voisin Maps

Voisin constructed self-rational maps of Calabi-Yau manifolds obtained as varieties of r-planes in cubic hypersurfaces of adequate dimension. This map has been thoroughly studied in the case $r = 1$, which is the Beauville-Donagi case. In this chapter, we compute the action of Ψ on holomorphic forms for any r. For $r = 2$, we compute the action of Ψ on the Chow group of 0-cycles, and confirm that it is as expected from the generalized Bloch conjecture.

4.1 Introduction

Let X be a smooth projective variety of dimension n defined over the field of complex numbers $\mathbb{C}$. The Chow group of k-cycles of X, denoted as $CH_k(X)$, is the quotient by the rational equivalence of the free abelian group generated by closed irreducible subvarieties of dimension k. In this work, we will ignore the complexities introduced by the torsion part of the Chow groups, focusing instead on $CH_k(X)_{\mathbb{Q}}$, which is the Chow group after tensoring with $\mathbb{Q}$.

Hodge theory emerges as a pivotal framework for investigating complex smooth projective varieties. The interaction between Hodge structures and the Chow groups is both enlightening and enigmatic. The cycle class map $cl : CH_k(X)_{\mathbb{Q}} \to H^{2n-2k}(X,\mathbb{Q})$ plays a central role in this interplay. A plethora of conjectures have been proposed, relating the complexity of Chow groups and that of Hodge structures. Among these, the Hodge conjecture stands out predicting that the image of cl coincides with the set of Hodge classes in $H^{2n-2k}(X,\mathbb{Q})$.

This chapter is primarily motivated by a conjecture put forth by Voisin, which we will explore in detail:

Conjecture 4.1.1 (Voisin)**.** *Let X be a strict Calabi-Yau manifold of dimension n. Let $C \subset CH_0(X)_{\mathbb{Q}}$ be the subgroup generated by the intersections of divisors and of Chern classes of X. Then the cycle class map*

$$cl : CH_0(X)_{\mathbb{Q}} \to H^{2n}(X,\mathbb{Q})$$

is injective on C.

Subsequent sections will delve into the terminology, underlying motivations, and the conceptual backdrop of Conjecture 4.1.1, setting the stage for a comprehensive exploration of its implications and relevance to the broader field of algebraic geometry.

4.1.1 Strict Calabi-Yau manifolds

Definition 4.1.2. A *strict Calabi-Yau manifold* is a simply connected compact Kähler manifold X of dimension at least 3, with trivial canonical bundle and such that $H^0(X,\Omega_X^k)=0$ for $0<k<\dim X$.

Remark 4.1.3. It follows from Kodaira's embedding theorem that such X is automatically projective.

Strict Calabi-Yau manifolds are pivotal in the study of algebraic varieties with trivial canonical bundles, known as K-trivial varieties. This class also includes hyper-Kähler manifolds, which are simply connected manifolds and possess a non-degenerate holomorphic 2-form, for which Conjecture 4.1.1 is also expected to be true when they are projective (this is the Beauville-Voisin conjecture [Bea07, Voi08]), and complex tori, which are defined as quotients of $\mathbb{C}^n$ by a lattice Γ, for which Conjecture 4.1.1 is definitely wrong, when they are projective.

Voisin's examples of Calabi-Yau manifolds

We focus on specific families of K-trivial varieties as constructed in [Voi04], generalizing the Beauville-Donagi construction [BD85]. Let $Y\subset\mathbb{P}^n$ be a smooth cubic hypersurface of dimension $n-1$, and let $r\geq 0$ denote a nonnegative integer. Define $X=F_r(Y)$ as the Hilbert scheme that parametrizes the r-dimensional linear subspaces in Y. As proved in [Voi04, (4.41)], for $n+1=\binom{r+3}{2}$ and a general Y, the variety X is a K-trivial variety of dimension $N=(r+1)(n-r)-\binom{r+3}{3}$. Specifically, when $r=0$, X is an elliptic curve; and as established in [BD85], X is a hyper-Kähler manifold for $r=1$. We have the following

Lemma 4.1.4. *For $r\geq 2$ and $n+1=\binom{r+3}{2}$, the constructed variety $X=F_r(Y)$ is a strict Calabi-Yau manifold.*

Proof. Reference [DM98, Proposition 3.1(a)] indicates that, generally, if $n\geq\frac{2}{r+1}\binom{r+3}{r}+r+1$, then $F_r(Y)$ is simply connected. This inequality is consistently satisfied for $r\geq 1$ when $n+1=\binom{r+3}{2}$. Consequently, X is simply connected in our context. The Beauville-Bogomolov decomposition theorem [Bea83, Théorème 1] gives a decomposition of $X\cong T\times W\times CY$, where T is a complex torus, W a product of hyper-Kähler manifolds, and CY a product of strict Calabi-Yau manifolds. As shown in [DM98, Théorème 3.4], the restriction morphism $H^i(\mathrm{Gr}(r+1,n+1),\mathbb{Q})\to H^i(F_r(Y),\mathbb{Q})$ is an isomorphism for $i<\min\{\dim F_r(Y),n-2r-1\}$. Under the condition $n+1=\binom{r+3}{2}$, $2<\min\{\dim F_r(Y),n-2r-1\}$ for any $r\geq 2$. Thus, $H^{2,0}(X)=H^{2,0}(\mathrm{Gr}(r+1,n+1))=0$, and the Picard number of X is 1 for $r\geq 2$. Hence, in the Beauville-Bogomolov decomposition of X, only the strict Calabi-Yau manifold component remains, and it is irreducible since the Picard number is 1. □

4.1.2 Conjectures on Chow groups

We begin by examining some foundational conjectures related to the Chow groups of smooth projective varieties. This discussion is then extended to include conjectures specifically concerning the Chow groups of projective hyper-Kähler and strict Calabi-Yau manifolds. These sections provide the necessary background and motivation for Conjecture 4.1.1, with our analysis closely following the insights and frameworks presented in [Voi04II], [Voi14], and [Voi16].

Bloch-Beilinson filtration and the generalized Bloch conjecture

Bloch and Beilinson have conjectured the existence of a descending filtration, denoted as $F^iCH_k(X)_{\mathbb{Q}}$, on the Chow groups with rational coefficients for any smooth complex projective variety X. This filtration is conjectured to satisfy a set of axioms.

Conjecture 4.1.5 (Bloch-Beilinson Conjecture [Voi04II]). *For every smooth projective variety X, there exists a descending filtration $F^\bullet$ on $CH^i(X)_{\mathbb{Q}}$ characterized by:*

(i) (Non-Triviality) $CH^i(X)_{\mathbb{Q}} = F^0CH^i(X)_{\mathbb{Q}}$, and $F^1CH^i(X)\mathbb{Q} = CH^i(X)_{\mathbb{Q},hom}$.

(ii) (Functoriality) For a cycle Z in $CH^k(X\times Y)_{\mathbb{Q}}$, the pushforward of $F^iCH^l(X)_{\mathbb{Q}}$ by Z_* is included within $F^iCH^{l+k-n}(Y)_{\mathbb{Q}}$, where n represents the dimension of X.

(iii) (Graded Component) If $[Z] = 0$ in $H^{2k}(X\times Y,\mathbb{Q})$, the induced map $Z_* : Gr^i_FCH^l(X)_{\mathbb{Q}} \to Gr^i_FCH^{l+k-n}(Y)_{\mathbb{Q}}$ vanishes for any i.

(iv) (Finiteness) The filtration terminates with $F^{k+1}CH^k(X)_{\mathbb{Q}} = 0$ for all varieties X and integers k.

The conjecture further suggests that the filtration of $CH_0(X)$ is intricately linked to the Hodge structures modulo Hodge substructures of coniveau ≥ 1, that is, to holomorphic forms.

Conjecture 4.1.6 (Generalized Bloch Conjecture for 0-Cycles). *Given a correspondence $Z \in CH^n(X\times Y)_{\mathbb{Q}}$ between smooth projective varieties X and Y, both of dimension n, if the map $[Z]^* : H^{i,0}(Y) \to H^{i,0}(X)$ vanishes for some $i \leq n$, then the pushforward $Z_* : Gr^i_FCH_0(X)_{\mathbb{Q}} \to Gr^i_FCH_{m-k}(Y)_{\mathbb{Q}}$ also vanishes for that i. Here, $F^\bullet$ represents the Bloch-Beilinson filtration and $Gr^\bullet_F$ its graded part.*

This conjecture is named the generalized Bloch conjecture as it extends the classical Bloch conjecture, stated as follows in [Blo80].

Conjecture 4.1.7 (Bloch [Blo80]). *For a correspondence $Z \in CH^2(S\times T)_{\mathbb{Q}}$ between surfaces that induces a null map $[Z]^* : H^{2,0}(T) \to H^{2,0}(S)$, the induced morphism $Z_* : F^2CH_0(S) \to F^2CH_0(T)$ is identically zero. Here, $F^2CH_0(S)$ is defined as the kernel of the Albanese map from $CH_0(S)_{hom}$ to $Alb(S)$, and similarly for $F^2CH_0(T)$.*

Conjectures of Beauville and Voisin

Let X be a projective hyper-Kähler manifold. Beauville's splitting conjecture, as introduced in [Bea07], suggests that the Bloch-Beilinson filtration on the Chow ring of X undergoes a natural multiplicative splitting. One weak version is now often referred to as the "weak splitting conjecture," detailed in the introduction of [Voi16].

Conjecture 4.1.8 (Beauville's Weak Splitting Conjecture [Bea07]). *For a projective hyper-Kähler manifold X, the cycle class map is injective on the subalgebra of $CH^*(X)$ that is generated by divisors.*

This conjecture was further expanded by Voisin in [Voi08] to include not only divisors but also Chern classes into the generating elements of the subalgebra.

Conjecture 4.1.9 (Voisin [Voi08])**.** *In the case of a projective hyper-Kähler manifold X, consider C^* to be the subalgebra of $CH^*(X)$ generated by divisors and Chern classes. The cycle class map is injective on C^*.*

However, generalizing these conjectures to strict Calabi-Yau manifolds often leads to contradictions. Following [Bea07, Examples 1.7], denote by Y the blow-up of $\mathbb{P}^3$ along a smooth curve of genus 2 and degree 5. Take a smooth divisor D in $|-2K_Y|$ and let X be the double covering of Y ramified along D, X is shown to be a strict Calabi-Yau threefold. However, the cycle class map $cl : CH_1(X)_{\mathbb{Q}} \to H^4(X,\mathbb{Q})$ fails to be injective on the subgroup generated by intersections of divisors, invaliding the "strict Calabi-Yau version" of Conjectures 4.1.8 and 4.1.9.

Despite these counterexamples concerning the "strict Calabi-Yau version" of Conjecture 4.1.9 for 1-cycles, it remains anticipated that the conjecture holds true for 0-cycles. This expectation takes the form of Conjecture 4.1.1, with compelling evidence provided by the constructions in [Baz17, Theorem 1.2].

Theorem 4.1.10 (Bazhov [Baz17])**.** *Let Y be a projective homogeneous variety of dimension $n+1 \geq 4$, and let X be a general element of the anti-canonical system $|-K_Y|$. Then X is a strict Calabi-Yau manifold satisfying Conjecture 4.1.1.*

4.1.3 Voisin maps

The distinct feature of the manifolds $X = F_r(Y)$ of Section 4.1.1 among all K-trivial manifolds revolves around the presence of a self-rational map, $\Psi : X \dashrightarrow X$, referred to as the Voisin map. This map was introduced in [Voi04] through the following construction: Consider a general point $x \in X$, representing an r-dimensional linear space P_x within Y. As demonstrated in [Voi04, Lemma 8], there exists a unique $(r+1)$-dimensional linear subspace H_x in $\mathbb{P}^n$ tangent to Y along P_x. The intersection $H_x \cap Y$ is a cubic hypersurface containing P_x doubly, leaving a residual r-dimensional linear subspace in Y represented by a point $x' \in X$. This process defines the Voisin map as $\Psi(x) = x'$.

The result from [AC08, Corollaire 2.2] indicates that for $r \geq 2$, the Voisin map does not preserve any non-trivial fibrations. Given this, exploring the dynamics of the Voisin map becomes a compelling avenue of study, although it falls outside the scope of this work. Notably, investigating the dynamics of the Voisin map in the hyper-Kähler case ($r = 1$) has yielded significant insights into the geometry of X, as reported in [Voi04, AV08, ABR11].

Our first result is a computation of two basic invariants of the map Ψ.

Theorem A. *Let $X = F_r(Y)$, $r \geq 0$ be as in 4.1.1, and let $\Psi : X \dashrightarrow X$ be the Voisin map. Then*

(i) For any $\omega \in H^0(X, K_X)$, we have

$$\Psi^* \omega = (-2)^{r+1} \omega.$$

(ii) The map $\Psi : X \dashrightarrow X$ is of degree 4^{r+1}.

Remark 4.1.11. Theorem A is trivial for $r = 0$. For $r = 1$, the results have been previously established in [Voi04].

Having Theorem A, Conjecture 4.1.6 leads us to the following

Conjecture 4.1.12. *Let $X = F_r(Y)$ with $r \geq 2$. Then for any $z \in CH_0(X)_{hom}$,*

$$\Psi_* z = (-2)^{r+1} z.$$

Indeed, let us explain how Conjecture 4.1.6 implies Conjecture 4.1.12. Let $N = \dim X$. As X is a strict Calabi-Yau manifold, we have $H^{k,0}(X) = 0$ for all $0 < k < N$. Consider $\Delta_X \in CH^n(X \times X)$ as the diagonal of $X \times X$. Then, for $0 < k < N$, $[\Delta_X]^*|_{H^{k,0}(X)} = 0$. Conjecture 4.1.6 implies that $\Delta_{X*}|_{Gr_F^k CH_0(X)} = 0$ for $0 < k < N$, leading to $Gr_F^k CH_0(X) = 0$ for these values of k. Thus, the Bloch-Beilinson filtration on $CH_0(X)$ simplifies to:

$$0 = F^{N+1}CH_0(X) \subseteq F^N CH_0(X) = \ldots = F^1 CH_0(X) = CH_0(X)_{hom} \subseteq F^0 CH_0(X) = CH_0(X).$$

Now, define $Z = \Gamma_\Psi - (-2)^{r+1}\Delta_X$. Given Theorem A, $[Z]^*|_{H^{N,0}(X)} = 0$. Applying Conjecture 4.1.6 once more, we find $Z_*|_{Gr_F^N CH_0(X)} = 0$, confirming that Z_* is null on $Gr_F^N CH_0(X) = F^N CH_0(X) = F^1 CH_0(X) = CH_0(X)_{hom}$, yielding the desired conclusion.

Our second main result is the proof of Conjecture 4.1.12 when $r = 2$

Theorem B. *Let $Y \subset \mathbb{P}^9$ be a general cubic 8-fold, and let $X = F_2(Y)$ be the Fano variety of planes in Y. Let $\Psi : X \dashrightarrow X$ be the Voisin map. Then for any $z \in CH_0(X)_{hom}$:*

$$\Psi_* z = -8z.$$

In the proof of Theorem B, the notion of constant-cycle subvarieties, as described in [Huy14, Voi16], plays a pivotal role. Let us revisit the definition for clarity.

Definition 4.1.13 ([Huy14, Voi16])**.** Let X be a smooth algebraic variety. A closed subvariety $j : Z \hookrightarrow X$ is a *constant-cycle subvariety* if every two points $z_1, z_2 \in Z$ are rationally equivalent in X. Equivalently, the image of the morphism, $j_* : CH_0(Z) \to CH_0(X)$, is $\mathbb{Z}$.

Let F denote the closure of the fixed locus under the Voisin map $\Psi : X \dashrightarrow X$. We next observe that Conjecture 4.1.12 immediately leads us to the following

Conjecture 4.1.14. *The variety $F \subset X$ is a constant-cycle subvariety for $r \geq 2$.*

Let us explain how Conjecture 4.1.14 is implied by Conjecture 4.1.12. Consider two points $x, y \in F$. Since the rational equivalence class is a countable union of closed algebraic subsets [Voi14, Section 1.1.1], we may assume that x, y are general. Assuming x, y are within the fixed locus of Ψ, we have $\Psi(x) = x$ and $\Psi(y) = y$, leading to $\Psi_*(x - y) = x - y$. However, Proposition 4.1.12 indicates $\Psi_*(x - y) = (-2)^r(x - y)$ under Conjecture 4.1.6. Consequently, $x - y = 0 \in CH_0(X)_{\mathbb{Q}}$. Roitman's theorem [Roi80] implies $CH_0(X)_{hom}$ is torsion-free due to the triviality of $Alb(X)$, so it follows that $x = y \in CH_0(X)$. Thus, $F \subset X$ forms a constant-cycle subvariety.

In the present paper, we will prove directly Conjecture 4.1.14 for $r = 2$, and this will be one step in our proof of Theorem B, that is, Conjecture 4.1.12 for $r = 2$.

Remark 4.1.15. Interestingly, even in the scenario of $r = 1$—not addressed in Conjecture 4.1.14—it has been proved [Voi08] that $F \subset X$ is a constant-cycle subvariety.

Now, let us examine the indeterminacy locus of the Voisin map $\Psi : X \dashrightarrow X$. This locus comprises two components, described as follows. The first component, Ind_0, contains points $x \in X$ that represent $P_x \subset Y$, such that there is more than one linear subspace of

dimension $r+1$ tangent to Y along P_x. The second component, Ind_1, includes points $x \in X$ representing $P_x \subset Y$, where an $(r+1)$-dimensional linear space exists that contains P_x and is contained within Y. It is not hard to prove (Proposition 4.4.4 and Proposition 4.4.5) that Ind_0 has codimension 2, while Ind_1 has codimension $r+2$ for $r \geq 2$ (and Ind_1 is empty for $r = 1$). We prove as a consequence of Theorem B the following result

Theorem C. *Assume $r = 2$. Then any 0-cycle of X which is a polynomial in the Chern classes of X and divisor classes is rationally equivalent to a cycle supported on Ind.*

Corollary 4.1.16. *If* $\mathrm{Ind} \subset X$ *is a constant cycle subvariety, Conjecture 4.1.1 holds true for* $X = F_2(Y)$.

This leaves us with an open

Question 4.1.17. Is $\mathrm{Ind} \subset X$ a constant cycle subvariety for $r \geq 2$?

Remark 4.1.18. In the hyper-Kähler case ($r = 1$), the indeterminacy locus Ind_0 is *not* a constant-cycle subvariety, nor is it Lagrangian, as indicated by [Ame09, Lemma 2] and [Voi04]. However, our understanding of why Ind_0 is not a constant-cycle subvariety in this scenario does not contradict the evidence - that we will present in Section 4.4.3 - for the positive answer to Question 4.1.17.

4.1.4 Notations and supplementary results

For the continuity and coherence of this chapter, we establish some notations that will be used throughout, unless specified otherwise:

- Y denotes a cubic hypersurface, which is typically smooth.
- X refers to a strict Calabi-Yau or hyper-Kähler manifold. Concrete examples of strict Calabi-Yau and hyper-Kähler manifolds in this chapter are constructed as the Fano variety of r-linear spaces in a smooth cubic hypersurface Y. The dimension of X is $(r+1)(n-r) - \binom{r+3}{3}$ and is denoted as N.
- P_x denotes the r-dimensional linear subspace in Y (resp. $\mathbb{P}^n$) for $x \in X$ (resp. $x \in \mathrm{Gr}(r+1, n+1)$).
- $\Psi : X \dashrightarrow X$ denotes the Voisin map.
- F represents the closure of the fixed locus of the Voisin map $\Psi : X \dashrightarrow X$.

Throughout the chapter, we establish several auxiliary results that not only support the proof of the main theorems but also hold intrinsic interest. Here is a summary of these findings:

- (Referencing Theorem 4.3.26 and Theorem 4.3.39): For $X = F_2(Y)$, the Fano variety of planes in a general cubic eightfold, it is established that $F \subset X$ is a constant cycle subvariety with a Chow class of $-404c_1^3 + 110c_1c_2 + 49c_3$.
- (Referencing Theorem 4.3.6): For a general cubic eightfold Y, it is demonstrated that $CH_1(F_1(Y))_{hom,\mathbb{Q}} = 0$.
- (Referencing Theorem 4.4.10): For the strict Calabi-Yau manifolds $X = F_r(Y)$, with $r \geq 2$, as presented in Section 4.1.1, the map $F_r : \mathbb{P}H^0(\mathbb{P}^n, \mathscr{O}_{\mathbb{P}^n}(3))/PSL_{n+1}(\mathbb{C}) \dashrightarrow \mathrm{Def}(X)$ is a local isomorphism.

4.2 Action of the Voisin map on top degree holomorphic forms

In this section, we compute the action $\Psi^* : H^0(X, K_X) \to H^0(X, K_X)$ of the Voisin map and its degree, proving Theorem A.

Remark 4.2.1. For $r = 0$, where Y is a plane cubic curve and $X = F_0(Y) = Y$ denotes an elliptic curve, $\Psi : X \to X$ acts by mapping x to $-2x$, in accordance with the addition law of the elliptic curve. Consequently, the degree of $\Psi : X \to X$ is 4. In the scenario where $r = 1$ and X is a hyper-Kähler fourfold, the result $\deg \Psi = 16$ is first discovered by Voisin [Voi04], and can be derived using either Chow-theoretic techniques [Voi04], [AV08, Corollary 1.7] or vector bundle methods [Huy24, Lemma 4.12 and Proposition 4.17].

Let us first note the following

Lemma 4.2.2. *The two assertions in Theorem A are equivalent, up to sign.*

Proof. (i) leads to (ii) as follows: Given that $\sigma \wedge \bar{\sigma}$ constitutes a volume form on X, it follows that

$$\deg \Psi \int_X \sigma \wedge \bar{\sigma} = \int_X \Psi^* \sigma \wedge \Psi^* \bar{\sigma} = \int_X (-2)^{r+1} \sigma \wedge (-2)^{r+1} \bar{\sigma} = 4^{r+1} \int_X \sigma \wedge \bar{\sigma}.$$

Conversely, let us demonstrate that (ii) implies $\Psi^* \sigma = \pm 2^{r+1} \sigma$, aligning closely with (i). The rational map can be consistently defined across the family of cubic hypersurfaces, implying the degree of Ψ remains invariant across different choices of the generic cubic hypersurface Y. Here, "generic" means that the cubic hypersurface is chosen outside a proper Zariski closed subset. Therefore, we may presume Y is defined over $\mathbb{Q}$. Consequently, the rational map $\Psi : X \dashrightarrow X$ is also defined over $\mathbb{Q}$. Thus, $\Psi^* : H^0(X, K_{X/\mathbb{Q}}) \to H^0(X, K_{X/\mathbb{Q}})$ operates by multiplication by a rational number, given that $H^0(X, K_{X/\mathbb{Q}})$ is a one-dimensional $\mathbb{Q}$-vector space. If $\Psi^* \sigma = \lambda \sigma$ with $\lambda \in \mathbb{Q}$, then $\lambda^2 = 4^{r+1}$, leading to $\lambda = \pm 2^{r+1}$. □

4.2.1 Proof of Theorem A (i)

Let $\mathrm{Fix}(\Psi) := \{x \in X : \Psi \text{ is defined at } x \text{ and } \Psi(x) = x\}$ be the fixed locus of $\Psi : X \dashrightarrow X$.

Proposition 4.2.3. *For Y general, the fixed locus* $\mathrm{Fix}(\Psi)$ *is not empty and is of codimension* $r+1$ *in* X.

Proof. We will suppose that $r \geq 2$ since the case $r = 1$ is shown in [ABR11, Proposition 3.1] already (see also [GK20, Corollary 3.13]). Let $B = \mathbb{P}H^0(\mathbb{P}^n, \mathscr{O}(3))$ be the parametrizing space of cubic hypersurfaces. Let $\mathrm{Fl} = \{(t,s) \in \mathrm{Gr}(r+1, n+1) \times \mathrm{Gr}(r+2, n+1) : P_t \subset \Pi_s\}$ be the flag variety of pairs of linear subspaces in $\mathbb{P}^n$. Let

$$\mathscr{I} = \{(f,t,s) \in B \times \mathrm{Fl} : \Pi_s \text{ is the only } \mathbb{P}^{r+1} \text{ such that } Y_f \cap \Pi_s = 3P_t\},$$

$$\tilde{\mathscr{I}} = \{(f,t,s) \in B \times \mathrm{Fl} : Y_f \cap \Pi_s \supset 3P_t\}.$$

Let $p : \mathscr{I} \to B$ (resp. $\tilde{p} : \tilde{\mathscr{I}} \to B$) and $q : \mathscr{I} \to \mathrm{Fl}$ (resp. $\tilde{q} : \tilde{\mathscr{I}} \to \mathrm{Fl}$) be the canonical projection maps. By definition of Ψ, for any $f \in B$, the fiber $p^{-1}(f) \subset \mathscr{I}$ coincides with the fixed locus of Ψ for the cubic hypersurface Y_f. Hence, it suffices to show that the map $p : \mathscr{I} \to B$ is dominant and that the dimension of the general fibers is $\dim X - r - 1$.

Lemma 4.2.4. *The variety $\mathscr{I}$ is open dense in $\tilde{\mathscr{I}}$. The dimension of $\mathscr{I}$ is* $\dim X + \dim H^0(\mathbb{P}^n, \mathscr{O}(3)) - r - 2$.

Proof. Let us consider the fibers of the map $\tilde{q} : \tilde{\mathscr{I}} \to \mathrm{Fl}$. For any element $(t,s) \in \mathrm{Fl}$ representing linear subspaces P and Π respectively, we may assume without loss of generality that

$$P = \{(x_0, x_1, \ldots, x_r, 0, \ldots, 0)\},$$
$$\Pi = \{(x_0, x_1, \ldots, x_r, x_{r+1}, 0, \ldots, 0)\}.$$

The fiber $\tilde{q}^{-1}((t,s))$ parametrizes the cubic hypersurfaces Y such that $\Pi \cap Y \supset 3P$. The last condition implies that the defining equation f of Y in $\mathbb{P}^n$ is given by

$$f(Y_0, \ldots, Y_n) = \alpha Y_{r+1}^3 + Y_{r+2}Q_{r+2} + \ldots + Y_nQ_n,$$

where $\alpha \in \mathbb{C}$ is a constant and $Q_{r+1}, \ldots, Q_n$ are quadratic polynomials. To write the above fact more formally, the fiber $\tilde{q}^{-1}((t,s))$ is identified with the image of the following map

$$\begin{array}{rccc} \Phi: & (\mathbb{C} \oplus H^0(\mathbb{P}^n, \mathscr{O}(2))^{\oplus n-r-1}) - \{0\} & \to & B \\ & (\alpha, Q_{r+1}, \ldots, Q_n) & \mapsto & \alpha Y_{r+1}^3 + Y_{r+2}Q_{r+2} + \ldots + Y_nQ_n. \end{array}$$

It is not hard to see that $\mathscr{I} \cap \tilde{q}^{-1}((t,s))$ parametrizes the cubic hypersurfaces Y whose defining equation f is given by

$$f(Y_0, \ldots, Y_n) = \alpha Y_{r+1}^3 + Y_{r+2}Q_{r+2} + \ldots + Y_nQ_n,$$

subject to $\alpha \neq 0$ and $Q_{r+1}(Y_0, \ldots, Y_r, 0, \ldots, 0), \ldots, Q_n(Y_0, \ldots, Y_r, 0, \ldots, 0)$ are linearly independent in $H^0(\mathbb{P}^r, \mathscr{O}(2))$. Notice that $n - r - 1 = \dim H^0(\mathbb{P}^r, \mathscr{O}(2))$, the above conditions give an open dense subset in $(\mathbb{C} \oplus H^0(\mathbb{P}^n, \mathscr{O}(2))^{\oplus n-r-1}) - \{0\}$, which implies that $\mathscr{I} \cap \tilde{q}^{-1}((t,s))$ is open dense in $\tilde{q}^{-1}((t,s))$ for any $(t,s) \in \mathrm{Fl}$. Therefore, $\mathscr{I}$ is open dense in $\tilde{\mathscr{I}}$.

For the next step, let us calculate the dimension of $\mathscr{I}$. To this end, we use now the projection $\mathscr{I} \to \mathrm{Fl}$ and compute the dimensions of its fibers. Since Fl is a $\mathbb{P}^{r+1}$-bundle over $\mathrm{Gr}(r+2, n+1)$, it is clear that $\dim \mathrm{Fl} = (r+2)(n-r-1) + (r+1)$. By the above description of the fiber $q^{-1}((t,s))$, the dimension of the fiber of $q : \mathscr{I} \to \mathrm{Fl}$ is the dimension of the space of cubic polynomials in $Y_0, \ldots, Y_n$ such that each monomial of it contains either one of the variables $Y_{r+2}, \ldots, Y_n$, and the latter is $\dim H^0(\mathbb{P}^n, \mathscr{O}(3)) - \dim H^0(\mathbb{P}^{r+1}, \mathscr{O}(3))$. Taken together, we have

$$\dim \mathscr{I} = (r+2)(n-r-1) + (r+1) + \dim H^0(\mathbb{P}^n, \mathscr{O}(3)) - \dim H^0(\mathbb{P}^{r+1}, \mathscr{O}(3)).$$

Taking into account of the relation $n+1 = \dim H^0(\mathbb{P}^{r+1}, \mathscr{O}(2))$ and the fact that $\dim X = (n-r)(r+1) - \dim H^0(\mathbb{P}^r, \mathscr{O}(3))$, we find that $\dim \mathscr{I} = \dim X + \dim H^0(\mathbb{P}^n, \mathscr{O}(3)) - r - 2$, as desired. □

Lemma 4.2.5. *The map $\tilde{p} : \tilde{\mathscr{I}} \to B$ is surjective.*

Proof. It is equivalent to showing that for any cubic hypersurface Y, there are linear subspaces $P, \Pi \subset \mathbb{P}^n$ of dimension r, $r+1$ respectively such that $\Pi \cap Y \supset 3P$. To construct such examples, it is worthwhile to notice that for $r \geq 2$, the variety $F_{r+1}(Y)$ is non-empty by a dimension counting argument. For any $\Pi \in F_{r+1}(Y)$ and any $P \subset \Pi$, we have $\Pi \cap Y \supset 3P$. That concludes the proof. □

By the two lemmas above, we conclude that the map $p : \mathscr{I} \to B$ is dominant. Since the map $p : \mathscr{I} \to B$ is dominant, the dimension of the general fiber is $\dim \mathscr{I} - \dim B$, which is equal to $\dim X - r - 1$. □

Proposition 4.2.6. *Let $x \in X$ be a generic fixed point of $\Psi : X \dashrightarrow X$ representing an r-dimensional linear subspace $P \subset Y$. Then the linear map*

$$\Psi_{*,x} : T_{X,x} \to T_{X,x}$$

has $N - r - 1$ eigenvalues equal to 1, corresponding to the tangent space to the fixed locus F of Ψ at x, and $r+1$ eigenvalues equal to -2, corresponding to the action of Ψ_* in the normal direction to F at x.

Proof. Since $x \in X$ is a generic fixed point of Ψ representing a plane $P \subset Y$, there is a unique $(r+1)$-plane $\Pi \subset \mathbb{P}^n$ such that $\Pi \cap Y = 3P$ as algebraic cycles. Without loss of generality, we may assume that

$$P = \{(x_0, x_1, \ldots, x_r, 0, \ldots, 0)\},$$

$$\Pi = \{(x_0, x_1, \ldots, x_r, x_{r+1}, 0, \ldots, 0)\}.$$

The fact $\Pi \cap Y = 3P$ implies that the defininig equation f of Y in $\mathbb{P}^n$ is given by

$$f(Y_0, \ldots, Y_n) = Y_{r+1}^3 + Y_{r+2}Q_{r+2} + \ldots + Y_nQ_n, \tag{4.1}$$

where $Q_{r+1}, \ldots, Q_n$ are quadratic polynomials. Let $v_1 \in T_{X,x}$ be a nonzero tangent vector given by a path $\{P_{1,t}\}_{t\in\Delta}$ with $P_{1,0} = P$, where Δ is the unit disc in the complex plane. To understand $\Psi_{*,x}(v_1) \in T_{X,x}$, we have to determine $P_{2,t} := \Psi(P_{1,t})$ for each small $t \in \Delta$. By the construction, for each small $t \in \Delta$, there is a unique $(r+1)$-plane $\Pi_t \subset \mathbb{P}^n$ such that $\Pi_t \cap Y = 2P_{1,t} + P_{2,t}$. Since the inclusions $P_{1,t} \subset \Pi_t \subset \mathbb{P}^n$ vary holomorphically with respect to $t \in \Delta$, we may assume that Π_t is given by the image of a linear map $G_t : \mathbb{P}^{r+1} \to \mathbb{P}^n$ varying holomorphically with $t \in \Delta$ and that

$$P_{1,t} = \{G_t(x_0, \ldots, x_r, 0)\}. \tag{4.2}$$

Then the relation $\Pi_t \cap Y = 2P_{1,t} + P_{2,t}$ can be translated as

$$f \circ G_t(x_0, \ldots, x_{r+1}) = x_{r+1}^2 L_t(x_0, \ldots, x_{r+1}) \tag{4.3}$$

for some linear function L_t and that

$$P_{2,t} = \{G_t(x_0, \ldots, x_{r+1}) : L_t(x_0, \ldots, x_{r+1}) = 0\}. \tag{4.4}$$

Let us simplify the notation and write $\mathbf{x} = (x_0, \ldots, x_{r+1})$. Let us write $G_t(\mathbf{x})$ as

$$G_t(\mathbf{x}) = (x_0 + tY_0(\mathbf{x}), x_1 + tY_1(\mathbf{x}), \ldots, x_{r+1} + tY_{r+1}(\mathbf{x}), tY_{r+2}(\mathbf{x}), \ldots, tY_n(\mathbf{x})) + O(t^2). \tag{4.5}$$

Then by (4.1), we have

$$f \circ G_t(\mathbf{x}) = x_{r+1}^3 + 3tx_{r+1}^2 Y_{r+1}(\mathbf{x}) + t \sum_{i=r+2}^{n} Y_i(\mathbf{x}) Q_i(\mathbf{x}, \vec{0}) + O(t^2). \tag{4.6}$$

Comparing (4.3) and (4.6), we get that x_{r+1}^2 divides $\sum_{i=r+2}^n Y_i(\mathbf{x})Q_i(\mathbf{x},\vec{0})$, and that

$$L_t(\mathbf{x}) = x_{r+1} + t\left\{3Y_{r+1}(\mathbf{x}) + \frac{Y_i(\mathbf{x})Q_i(\mathbf{x},\vec{0})}{x_{r+1}^2}\right\} + O(t^2). \tag{4.7}$$

Hence, the equation $L_t(\mathbf{x}) = 0$ has an explicit expression as

$$x_{r+1} = -t\left\{3Y_{r+1}(\mathbf{x}',0) + \frac{\sum_{i=r+2}^n Y_iQ_i}{X_{r+1}^2}(\mathbf{x}',0)\right\} + O(t^2), \tag{4.8}$$

where $\mathbf{x}' = (x_0,\ldots,x_r)$. Comparing (4.4)(4.5)(4.8), we get

$$\begin{aligned} P_{2,t} =&\{(x_0 + tY_0(\mathbf{x}',0),\ldots,x_r + tY_r(\mathbf{x}',0), -2tY_{r+1}(\mathbf{x}',0) - t\frac{\sum Y_iQ_i}{X_{r+1}^2}(\mathbf{x},0),\\ &tY_{r+2}(\mathbf{x}',0),\ldots,tY_n(\mathbf{x}',0)) + O(t^2)\}. \end{aligned}$$

We view now $v_1 \in T_{X,x}$ as a $(r+1)\times(n-r)$-matrix via the following natural inclusion and identification $T_{X,x} \subset T_{\mathrm{Gr}(r+1,n+1),x} = Hom(\mathbb{C}^{r+1}, \mathbb{C}^{n+1}/\mathbb{C}^{r+1})$. Then v_1 is represented by the matrix

$$\begin{pmatrix} Y_{r+1} \\ Y_{r+2} \\ \vdots \\ Y_n \end{pmatrix},$$

where we view each Y_i as a row vector $(a_0,a_1,\ldots,a_r)$ if $Y_i(x_0,\ldots,x_r,0) = a_0x_0 + \ldots + a_rx_r$. By the above calculations, we see that $\Psi_{*,x}(v_1)$ is represented by the matrix

$$\begin{pmatrix} -2Y_{r+1} - \frac{\sum_{i=r+2}^n Y_iQ_i}{X_{r+2}^2} \\ Y_{r+2} \\ \vdots \\ Y_n \end{pmatrix}$$

with the same explanation about the notations. Hence, $\Psi_{*,P}$ sends $\begin{pmatrix} Y_{r+1} \\ Y_{r+2} \\ \vdots \\ Y_n \end{pmatrix}$ to $\begin{pmatrix} -2Y_{r+1} - \frac{\sum_{i=r+2}^n Y_iQ_i}{X_{r+2}^2} \\ Y_{r+2} \\ \vdots \\ Y_n \end{pmatrix}$. The representing matrix of $\Psi_{*,x}$ is uppper triangular with the diagonal $r+1$ copies of -2 and $N-r-1$ copies of 1. Therefore, the eigenpolynomial of $\Psi_{*,x} : T_{X,x} \to T_{X,x}$ is $(t+2)^{r+1}(t-1)^{N-r-1}$, as desired. □

Proof of Theorem A (i). By Proposition 4.2.6, the eigenpolynomial of $\Psi_P^* : \Omega_{X,P} \to \Omega_{X,P}$ is $(t+2)^{r+1}(t-1)^{N-r-1}$ for a generic fixed point P of Ψ. Hence, the map $\Psi_P^* : K_{X,P} \to K_{X,P}$ is given as the multiplication by $(-2)^{r+1}$. Let $\omega \in H^0(X,K_X)$ be a nowhere zero top degree holomorphic differential form on X. Then, as $H^0(X,K_X)$ has dimension 1,

$$\Psi^*\omega = \lambda\omega \tag{4.9}$$

for some $\lambda \in \mathbb{C}$. By evaluating the equation (4.9) at the point $P \in X$, we find $\lambda = (-2)^{r+1}$, and we finish the proof of Theorem A (i). □

4.2.2 A direct proof of Theorem A (ii)

While Theorem A (ii) has been established as a consequence of Theorem A (i) using Lemma 4.2.2, we present a direct proof, using an enumerative geometry viewpoint.

Proof of Theorem A (ii). The beginning of the proof is similar to that of [Huy24, Lemma 4.12]. Let $P' \in X$ be a generic r-plane in Y. The preimage of P' via Ψ is the set of r-planes P in Y such that there is an $(r+1)$-plane Π in $\mathbb{P}^n$ such that $\Pi \cap Y = 2P + P'$ as algebraic cycles. In general, let $\Pi \subset \mathbb{P}^n$ be an $(r+1)$-plane containing P'. Then $\Pi \cap Y = Q \cup P'$ where $Q \subset \Pi$ is a quadratic hypersurface that corresponds to a quadratic form q_Π. The preimage of P' via Ψ corresponds to the $(r+1)$-planes Π such that the quadratic form q_Π is of rank 1. Now let $\pi : \mathbb{P}^n \dashrightarrow \mathbb{P}^{n-r-1}$ be the projection map induced by the r-plane P'. One can resolve the indeterminacies of $\pi : \mathbb{P}^n \dashrightarrow \mathbb{P}^{n-r-1}$ by blowing up $\mathbb{P}^n$ along P'. Let $p : \mathscr{P} \to \mathbb{P}^n$ be the blowup map. Then the induced map $q : \mathscr{P} \to \mathbb{P}^{n-r-1}$ is a projective bundle induced by a vector bundle $\mathscr{E}$ over $\mathbb{P}^{n-r-1}$. We now prove

Lemma 4.2.7. *(i) We have* $\mathscr{E} \cong \mathscr{O}_{\mathbb{P}^{n-r-1}}^{r+1} \oplus \mathscr{O}_{\mathbb{P}^{n-r-1}}(-1)$.
(ii) There is a section $q \in H^0(\mathbb{P}^{n-r-1}, Sym^2\mathscr{E}^* \otimes \mathscr{O}_{\mathbb{P}^{n-r-1}}(1))$ *such that on each point* $\Pi \in \mathbb{P}^{n-r-1}$*,* q *coincides with* q_Π.

Proof. See [Huy24, Section 1.5.1-1.5.2]. □

Hence, to calculate the degree of $\Psi : X \dashrightarrow X$, we need to calculate the number of points in $\mathbb{P}^{n-r-1}$ over which q is of rank 1. Since P' is generic, by a dimension counting argument we find that there is no plane Π containing P' that is contained in Y. Therefore, we are reduced to calculate the degree of the locus where a generic section of $Sym^2\mathscr{E}^* \otimes \mathscr{O}(1)$ has rank ≤ 1. Let $f : \mathbb{P}^{n-r-1} \to \mathbb{P}^{n-r-1}$ be a morphism of degree 2^{n-r-1} (e.g., $f : (x_0, \ldots, x_{n-r-1}) \mapsto (x_0^2, \ldots, x_{n-r-1}^2)$). Let $Z_q \subset \mathbb{P}^{n-r-1}$ be the locus where $q \in H^0(\mathbb{P}^{n-r-1}, Sym^2\mathscr{E}^* \otimes \mathscr{O}(1))$ is of rank ≤ 1. Then f^*Z_q is the locus where $f^*q \in H^0(Sym^2\mathscr{F})$ is of rank ≤ 1. Here, $\mathscr{F} = \mathscr{O}(1)^{r+1} \oplus \mathscr{O}(3)$. Then

$$\deg f^*Z_q = \deg f \cdot \deg Z_q = 2^{n-r-1} \deg Z_q. \tag{4.10}$$

Let $p : \mathbb{P}(\mathscr{F}) \to \mathbb{P}^{n-r-1}$ and $\pi : \mathbb{P}(\mathrm{Sym}^2\mathscr{F}) \to \mathbb{P}^{n-r-1}$ be the projective bundles over $\mathbb{P}^{n-r-1}$ corresponding to the vector bundles $\mathscr{F}$ and $\mathrm{Sym}^2\mathscr{F}$, respectively. Consider the Veronese map of degree 2 over the projective space $\mathbb{P}^{n-r-1}$.

$$\begin{array}{rccc} v_2 : & \mathbb{P}(\mathscr{F}) & \to & \mathbb{P}(\mathrm{Sym}^2\mathscr{F}) \\ & \alpha & \mapsto & \alpha^2. \end{array}$$

The section f^*q induces a section σ_q of the projective bundle $\pi : \mathbb{P}(Sym^2\mathscr{F}) \to \mathbb{P}^{n-r-1}$ such that f^*Z_q coincides with $\pi_*(\mathrm{Im}\sigma_q \cap \mathrm{Im}(v_2)) = p_*(v_2^*(\mathrm{Im}(\sigma_q))$. Let $\mathscr{S}$ (resp. $\mathscr{S}'$) be the tautological subbundle of $\mathbb{P}(\mathscr{F})$ (resp. $\mathbb{P}(Sym^2\mathscr{F})$). Let $\mathscr{Q}'$ be the tautological quotient bundle of $\mathbb{P}(Sym^2\mathscr{F})$. Then the pull-back of $f^*q \in H^0(\mathbb{P}^{n-r-1}, Sym^2\mathscr{F})$ gives a section of $H^0(\mathbb{P}(Sym^2\mathscr{F}), \mathscr{Q}')$ whose zero locus coincides with $\mathrm{Im}(\sigma_q)$. Writing $c(\mathscr{Q}')$ as $\pi^*c(\mathrm{Sym}^2\mathscr{F}) \cdot c(\mathscr{S}')^{-1}$ and noticing that $v_2^*\mathscr{S}' = \mathscr{S}^{\otimes 2}$, we see that f^*Z_q is the part of degree n of $p_*(c(S^{\otimes 2})^{-1} \cdot p^*c(\mathrm{Sym}^2\mathscr{F}))$. Let $s(\mathscr{F})$ be the formal series of the Segre classes

of $\mathscr{F}$ [Ful84, Chapter 3] and let $h = c_1(\mathscr{O}_{\mathbb{P}^n}(1))$. By a direct calculation, we find that

$$\begin{aligned} f^*Z_q &= 2^{r+1}\sum_i 2^i s_i(\mathscr{F})\cdot c_{n-r-1-i}(\mathrm{Sym}^2\mathscr{F}) \\ &= \text{The degree } n-r-1 \text{ part of } 2^{r+1}\frac{(1+2h)^{\frac{(r+1)(r+2)}{2}}\cdot(1+4h)^{r+1}\cdot(1+6h)}{(1+2h)^{r+1}\cdot(1+6h)} \\ &= 2^{r+1}\cdot 2^{\frac{(r+1)r}{2}}\cdot 4^{r+1}h^{n-r-1}. \end{aligned}$$

Taking into account of the relation (4.10), we find that $\deg Z_q = 4^{r+1}$, as desired. □

4.3 Action of the Voisin map on $CH_0(X)_{hom}$

The main purpose of this section is to prove Theorem B. Part of the argument will work for any r and will be given in Section 4.3.1.

4.3.1 Decomposition of the action of Ψ_*

Let $X = F_r(Y)$ be as presented in Section 4.1.1. Let $I_r = \{(x,x') \in X\times X : \dim(P_x \cap P_{x'}) \geq r-1\}$.

Theorem 4.3.1. *There is an $(r+1)$-cocycle $\gamma \in CH^{r+1}(X)$ lying in the image of the restriction map $CH^{r+1}(\mathrm{Gr}(r+1,n+1)) \to CH^{r+1}(X)$ such that for any $z \in CH_0(X)$, we have in $CH_0(X)_{\mathbb{Q}}$*

$$\Psi_* z = (-2)^{r+1}z + \gamma\cdot I_{r*}(z). \tag{4.11}$$

Proof. The argument is quite similar to that in the proof of [Voi14, Theorem 4.16]. Let $B = \mathbb{P}H^0(\mathbb{P}^n,\mathscr{O}(3))$ be the projective space parametrizing all cubic hypersurfaces in $\mathbb{P}^n$. For $f \in B$, we denote $Y_f \subset \mathbb{P}^9$ the hypersurface defined by f. There is a universal Fano variety of r-spaces defined as follows.

$$\mathscr{X} := \{(f,x) \in B\times \mathrm{Gr}(r+1,n+1) : P_x \subset Y_f\}.$$

The fiber of the projection map $\pi : \mathscr{X} \to B$ over a point $f \in B$ is the Fano variety of r-spaces of Y_f. The Voisin map $\Psi : X \dashrightarrow X$ can also be defined universally. In fact, we define the universal graph $\Gamma_{\Psi_{\mathrm{univ}}}$ as the closure of the set of pairs $(x,x') \in \mathscr{X}\times_B\mathscr{X}$ such that there exists a $(r+1)$-dimensional linear subspace $H_{r+1} \subset \mathbb{P}^n$ such that $H_{r+1}\cap Y_{\pi(x)} = 2P_x + P_{x'}$. Denote

$$\begin{array}{ccccc} \Gamma_{\Psi_{\mathrm{univ}}} & \xhookrightarrow{i} & \mathscr{X}\times_B\mathscr{X} & \xrightarrow{q} & \mathrm{Gr}(r+1,n+1)\times\mathrm{Gr}(r+1,n+1) \\ & & \downarrow p & & \\ & & B & & \end{array}$$

the inclusion map and the natural projection maps. Then over a point $f \in B$ corresponding to a general smooth cubic hypersurface Y_f, the fiber $(p\circ i)^{-1}(f)$ is the graph of the Voisin

map on $F_r(Y_f)$. There is a stratification of $\mathrm{Gr}(r+1,n+1)\times\mathrm{Gr}(r+1,n+1)$ given as follows.

$$\begin{array}{c}\mathrm{Gr}(r+1,n+1)\times\mathrm{Gr}(r+1,n+1)\\ \uparrow\\ I_1^G:=\{(x,x'):P_x\cap P_{x'}\neq\emptyset\}\\ \uparrow\\ I_2^G:=\{(x,x'):P_x\cap P_{x'}\text{ contains a line }\}\\ \uparrow\\ \vdots\\ \uparrow\\ I_r^G:=\{(x,x'):P_x\cap P_{x'}\text{ contains an }(r-1)\text{-space}\}\\ \uparrow\\ I_{r+1}^G=\Delta_G:=\{(x,x)\}\end{array}$$

In other words, the subvariety I_k^G is defined as

$$I_k^G=\{(x,x')\in\mathrm{Gr}(r+1,n+1)\times\mathrm{Gr}(r+1,n+1):\dim(P_x\cap P_{x'})\geq k-1\}.$$

The map $q:\mathscr{X}\times_B\mathscr{X}\to\mathrm{Gr}(r+1,n+1)\times\mathrm{Gr}(r+1,n+1)$ has a projective bundle structure over each stratum. Precisely, let $d=\dim B$. Let $I_k^{\mathscr{X}}$ be the preimage of I_k^G under the map $q:\mathscr{X}\times_B\mathscr{X}\to\mathrm{Gr}(r+1,n+1)\times\mathrm{Gr}(r+1,n+1)$. Over the open subset $I_k^G-I_{k+1}^G$, the map $q_{|I_k^{\mathscr{X}}}$ is a $\mathbb{P}^{d-\delta_k}$-bundle where $\delta_k=2h^0(\mathbb{P}^r,\mathscr{O}(3))-h^0(\mathbb{P}^{k-1},\mathscr{O}(3))$. Let $i_k^{\mathscr{X}}:I_k^{\mathscr{X}}\hookrightarrow\mathscr{X}\times_B\mathscr{X}$ be the inclusion maps. The Chow ring of $\mathscr{X}\times_B\mathscr{X}$ is thus equal to

$$\bigoplus_i h^i\cdot\left(\bigoplus_k\left(q_{|I_k^{\mathscr{X}}}\right)^*CH^*(I_k^G)\right),$$

where $h=p^*\mathscr{O}_B(1)$. Now let us consider the Chow class of $\Gamma_{\Psi_{\mathrm{univ}}}$ in $CH^N(\mathscr{X}\times_B\mathscr{X})$, where $N=(r+1)(n-r)-\binom{r+3}{3}$ is the dimension of X. By the construction, we have $\Gamma_{\Psi_{\mathrm{univ}}}\subset I_r^{\mathscr{X}}$, thus

$$\Gamma_{\Psi_{\mathrm{univ}}}\in\bigoplus_i h^i\cdot\left(i_{r+1,*}^{\mathscr{X}}(q_{|\Delta_{\mathscr{X}}})^*CH^{-i}(\Delta_G)+i_{r*}^{\mathscr{X}}\left(q_{|I_k^{\mathscr{X}}}\right)^*CH^{m-i}(I_r^G)\right),$$

where m is the relative dimension of $pr_1:I_r\to X$ that can be calculated as follows. Over a point $x\in X$, the fiber of $pr_1:I_r\to X$ is the set of points $x'\in X$ such that the intersection $P_x\cap P_{x'}$ contains a $\mathbb{P}^{r-1}$ in P_x. The set of $\mathbb{P}^{r-1}$ contained in P_x is a $\mathbb{P}^r$. Furthermore, for each given $\mathbb{P}^{r-1}\subset Y$, the set of r-spaces $P_{x'}\subset\mathbb{P}^n$ that contain the given $\mathbb{P}^{r-1}$ is a $\mathbb{P}^{n-r}$, and the condition that $P_{x'}\subset Y$ is equivalent to saying that the defining equation of the residual quadric is identically zero, which gives $\frac{(r+1)(r+2)}{2}$ independent conditions. Taking everything into account, we find that $m=n-\frac{(r+1)(r+2)}{2}=r+1$.

Now since Γ_Ψ is a fiber of $p\circ i:\Gamma_{\Psi_{\mathrm{univ}}}\to B$, we conclude that

$$\Gamma_\Psi=\Gamma_{\Psi_{\mathrm{univ}}|X\times X}\in(i_{r+1,*}^{\mathscr{X}}q^*CH^0(\Delta_G)+i_{r*}^{\mathscr{X}}q^*CH^{r+1}(I_r^G))|_{X\times X}.$$

Therefore, we can write

$$\Gamma_\Psi = \alpha\Delta_X + \delta, \tag{4.12}$$

where $\alpha \in \mathbb{Z}$ is a coefficient and $\delta \in (i^{\mathscr{X}}_{r*} q^* CH^{r+1}(I_r^G))|_{X\times X}$. Write $\delta = i^{\mathscr{X}}_{r*}(q^*\delta_G)$ for some $\delta_G \in CH^{r+1}(I_r^G)$. Notice that I_r^G is a fiber bundle over $\mathrm{Gr}(r+1,n+1)$ whose fiber is a closed Schubert subvariety Σ of $\mathrm{Gr}(r+1,n+1)$, that is defined as the closure of the set of r-spaces in $\mathbb{P}^n$ that intersects with a given r-space along a $(r-1)$-space. This fiber bundle has a universal cellular decomposition into affine bundles in the sense of [Ful84, Example 1.9.1]. By [Ful84, Example 19.1.11] and [Voi03, Théorème 7.33], we have

$$CH^*(I_r^G) = CH^*(\mathrm{Gr}(r+1,n+1)) \otimes CH^*(\Sigma).$$

Therefore, we can write

$$\delta_G = \sum_{i=0}^{r+1} (p_1^*\alpha_i).\beta_i,$$

where $\alpha_i \in CH^i(\mathrm{Gr}(r+1,n+1)$ and $\beta_i \in CH^{r+1-i}(\Sigma)$. One easily checks that the morphism $pr_2^* : CH^*(Gr(r+1,n+1)) \to CH^*(\Sigma)$ is surjective. Hence, there exists $\gamma_i \in CH^{r+1-i}(\mathrm{Gr}(r+1,n+1))$ such that $\beta_i = pr_2^*\gamma_i$. Therefore,

$$\delta_G = \sum_{i=0}^{r+1} p_1^*\alpha_i.p_2^*\gamma_i. \tag{4.13}$$

Now we prove

Lemma 4.3.2. *The coefficient α in the decomposition (4.12) equals $(-2)^{r+1}$.*

Proof. For $\omega \in H^{N,0}(X)$, we have $\delta^*\omega = p_{2*}(\sum_i p_1^*([\alpha_i]\cup\omega)\cup\gamma_{i|X}\cup[I_r])$. Since ω is a top degree form, $[\alpha_i]\cup\omega = 0$ unless $i=0$ and thus, $\delta^*\omega = c[\gamma_{0|X}]\cup I_r^*\omega$, where c is some constant number. Now let us prove that $I_r^*\omega = 0$. Let $\mathscr{P}_{r,r-1} = \{(x,\lambda) \in F_r(Y)\times F_{r-1}(Y) : P_\lambda \subset P_x\}$ be the flag variety. Since $I_r = \mathscr{P}^t_{r,r-1}\circ\mathscr{P}_{r,r-1}$ as correspondences, we have $I_r^*\omega = \mathscr{P}^*_{r,r-1}(\mathscr{P}_{r,r-1*}(\omega)) = 0$ since $\mathscr{P}_{r,r-1*}\omega \in H^{N-1,-1}(F_1(Y))$. Therefore, $\Psi^*\omega = \alpha\omega + \delta^*\omega = \alpha\omega$. By Theorem A, we get $\alpha = (-2)^{r+1}$. □

Since $z \in CH_0(X)_\mathbb{Q}$ is a 0-cycle, $\delta_* z = p_{2*}(\sum_i p_1^*(\alpha_i\cdot z)\cdot p_2^*\gamma_{i|X}\cdot I_r) = p_{2*}(p_1^*(\alpha_0\cdot z)\cdot p_2^*\gamma_{0|X}\cdot I_r) = \gamma\cdot I_{r*}z$ where γ is some multiple of $\gamma_{0|X}$, which is an element in the image of the restriction map $CH^{r+1}(\mathrm{Gr}(r+1,n+1)) \to CH^{r+1}(X)$. Taking Lemma 4.3.2, formula (4.12) and formula (4.13) into account, we prove the formula

$$\Psi_* z = (-2)^{r+1} z + \gamma\cdot I_{r*}z, \tag{4.14}$$

as desired. □

The cycle gamma appearing in (4.14) is a polynomial in the Chern classes c_i of the tautological bundle of the Grassmannian, restricted to X. Let us now prove

Proposition 4.3.3. *We have $c_{r+1}\cdot I_{r*}z$=0 for $z \in CH_0(X)_{hom}$.*

Proof. Let $H_{n-1} \subset \mathbb{P}^n$ be a general hyperplane. Let us define the Schubert variety $\Sigma_{r+1}^{H_{n-1}}$ as

$$\Sigma_{r+1}^{H_{n-1}} := \{y \in \mathrm{Gr}(r+1,n+1) : P_y \subset H_{n-1}\}.$$

Then c_{r+1} is represented by the class of $\Sigma_{r+1}^{H_{n-1}}$. Let $x \in X$ be a general point and let

$$\Theta_x := \{y \in X : \dim(P_y \cap P_x) \geq r-1\}.$$

Then by the definition of I_r, Θ_x represents $I_{r*}x$. By definition, the intersection $\Sigma_{r+1}^{H_{n-1}} \cap \Theta_x$ is the set

$$\{y \in X : P_y \subset H_{n-1} \cap Y, \dim P_x \cap P_y \geq r-1\},$$

which we can also rewrite, if P_x is not contained in H_{n-1}, as

$$\Sigma_{r+1}^{H_{n-1}} \cap \Theta_x = \{y \in X : Y \cap H_{n-1} \supset P_y \supset P_x \cap H_{n-1}\}.$$

Let $\Delta_x = P_x \cap H_{n-1}$. Then Δ_x provides a point δ_x of the variety $F_{r-1}(Y \cap H_{n-1})$. Let $i : F_r(Y \cap H_{n-1}) \hookrightarrow X$ be the natural embedding map. Let $\mathscr{P}_{r,r-1}^{H_{n-1}} := \{(y,\lambda) \in F_r(Y \cap H_{n-1}) \times F_{r-1}(Y \cap H_{n-1}) : P_\lambda \subset P_y\}$ be the incidence variety. By the above description of $\Sigma_{r+1}^{H_{n-1}} \cap \Theta_x$, one finds that

$$c_{r+1} \cdot I_{r*}x = i_*(\mathscr{P}_{r,r-1}^{H_{n-1}*}(\delta_x)),$$

where Δ_x is viewed as an element in $CH_0(F_{r-1}(Y \cap H_{n-1}))$. One can verify that $F_{r-1}(Y \cap H_{n-1})$ is a Fano manifold and thus, $CH_0(F_{r-1}(Y \cap H_{n-1})) = \mathbb{Z}$. Therefore, the Chow class of $c_{r+1} \cdot I_{r*}x$ does not depend on x. That is, for any $z \in CH_0(X)_{hom}$, we get $c_{r+1} \cdot I_{r*}z = 0$ in $CH_0(X)$, as desired. □

Proposition 4.3.4. *If $CH_1(F_{r-1}(Y))_{\mathbb{Q}}$ is trivial, then $c_r \cdot I_{r*}(z) = 0$ in $CH_1(X)$ for any $z \in CH_0(X)_{hom}$.*

Proof. We only write down the case $r = 2$, for which the assumption of Proposition 4.3.4 will be proved in the next section. The general case is similar. Let H_7 be a 7-dimensional linear subspace in $\mathbb{P}^9$. In the Grassmannian $\mathrm{Gr}(3,10)$, the class c_2 is represented by the subvariety $\Sigma_2^{H_7}$ defined as

$$\Sigma_2^{H_7} := \{y \in \mathrm{Gr}(3,10) : P_y \cap H_7 \text{ contains a line } \Delta_{y,H}\}.$$

For general $x \in X$, the plane P_x intersects H_7 at a single point z_x, and the class $I_{2,*}x$ is represented by the following subvariety Θ_x in X defined as

$$\Theta_x := \{y \in X : P_y \cap P_x \text{ contains a line } \Delta_{xy}\}.$$

Let $H_6 \subset H_7$ be a linear subspace of dimension 6 not containing the point z_x.

Lemma 4.3.5. *For general $x \in X$, and any $y \in \Theta_x$, we have $z_x \in \Delta_{xy}$.*

Proof. Let P_y be a plane in Y that intersects P_x along the line Δ_{xy} and intersects H_7 along the line $\Delta_{y,H}$. Then Δ_{xy} and $\Delta_{y,H}$ must intersect, since they are two lines in a projective plane; and the intersection point of Δ_{xy} and $\Delta_{y,H}$ must be z_x since z_x is the only intersection point of P_x and H_7. Therefore, $z_x \in \Delta_{xy}$, as desired. □

Let $\Sigma_1^{H_6} := \{y \in \mathrm{Gr}(3,10) : P_y \cap H_6 \neq \emptyset\}$. Let Ξ_x be the variety of points $y \in X$ such that $P_y \cap P_x$ contains a line Δ_{xy} containing z_x. By Lemma 4.3.5, we now conclude that

$$\Sigma_2^{H_7} \cap \Theta_x = \Sigma_1^{H_6} \cap \Xi_x. \tag{4.15}$$

Indeed, by Lemma 4.3.5, $\Sigma_2^{H_7} \cap \Theta_x$ contains all the points $y \in X$ such that $P_y \cap P_x$ contains a line Δ_{xy} containing z_x and such that $P_y \cap H_7$ contains a line $\Delta_{y,H}$. This variety coincides with $\Sigma_1^{H_6} \cap \Xi_x$ since knowing that $z_x \in P_y$ and that $z_x \notin H_6$, $P_y \cap H_7$ contains a line if and only if $P_y \cap H_6$ is nonempty.

Let $\mathscr{P}_{2,1} := \{(x', [l]) \in F_2(Y) \times F_1(Y) : l \subset P_{x'}\}$ be the flag variety. Let $\Delta^\vee_{P_x,z_x} = \{[l] \in F_1(Y) : z_x \in l \subset P_x\}$. Then $\Delta^\vee_{P_x,z_x}$ provides a class $\delta^\vee_{P_x,z_x} \in CH_1(F_1(Y))$. The equation (4.15) shows that $c_2 \cdot I_{2,*}x = c_1 \cdot \mathscr{P}^*_{2,1}(\delta^\vee_{P_x,z_x})$. Therefore, for $z \in CH_0(X)_{\mathbb{Q},hom}$, we have $c_2 \cdot I_{2,*}z = c_1 \cdot \mathscr{P}^*_{2,1}(Z)$ for some $Z \in CH_1(F_1(Y))_{\mathbb{Q},hom}$. But we have $CH_1(F_1(Y))_{\mathbb{Q},hom} = 0$ by Theorem 4.3.6 below (or by assumption for $r > 2$). Hence, $c_2 \cdot I_{2,*}z = 0 \in CH_1(X)_{\mathbb{Q}}$, as desired. □

4.3.2 Triviality of $CH_1(F_1(Y))_{\mathbb{Q}}$

Let $Y \subset \mathbb{P}^9$ be a cubic eightfold. It has been established that $H^{p,q}(F_1(Y)) = 0$ for $p \leq 1$ and $p \neq q$ [DM98], indicating that the coniveau of $F_1(Y)$ is at least 2. According to the generalized Bloch conjecture, this suggests that $CH_i(F_1(Y))_{\mathbb{Q},hom} = 0$ for $i \leq 1$. This section is devoted to proving this statement, namely

Theorem 4.3.6. *The Chow group $CH_1(F_1(Y))_{\mathbb{Q}}$ is isomorphic to $\mathbb{Q}$.*

Lines of lines

Let P be a plane contained in Y and let $x \in P$ be a point. Let $\Delta^\vee_{P,x}$ be the variety of lines in P passing through x. Viewed as a 1-cycle of $F_1(Y)$, it is clear that the Chow class of $\Delta^\vee_{P,x}$ does not depend on the choice of $x \in P$ and we let $\Delta^\vee_P \in CH_1(F_1(Y))$ denote the Chow class of $\Delta^\vee_{P,x}$ for some (and thus any) $x \in P$.

Proposition 4.3.7. *Let $L \cong \mathbb{P}^3 \subset \mathbb{P}^9$ be a 3-dimensional linear subspace whose intersection with Y contains three planes P_1, P_2 and P_3 (not necessarily distinct). Then the Chow class*

$$\Delta^\vee_{P_1} + \Delta^\vee_{P_2} + \Delta^\vee_{P_3} \in CH_1(F_1(Y))$$

does not depend on the choice of L.

Remark 4.3.8. The linear spaces $L \cong \mathbb{P}^3$ whose intersection with Y is the union of three planes form a projective variety of general type, which is not rationally connected. To prove the proposition, we consider a larger space, namely the variety of cubic surfaces in Y which are cones and prove that this space is rationally connected.

Proof of Proposition 4.3.7. Let $\mathscr{M}$ be the space of cubic surfaces in Y which are cones. An element of $\mathscr{M}$ is of the form (S, y_0) where S is a cubic surface that is a cone from a vertex y_0. If P_1, P_2, P_3 are planes as in the Proposition, and y_0 is a point in the intersection of the three planes, then $P_1 \cup P_2 \cup P_3$ is a cone with vertex y_0, hence $(P_1 \cup P_2 \cup P_3, y_0)$ is an element of $\mathscr{M}$. Consider the incidence variety

$$\mathscr{E} = \{((S, y_0), l) \in \mathscr{M} \times F_1(Y) : l \text{ is a line in } S \text{ passing through } y_0\}.$$

Then $\Delta^\vee_{P_1} + \Delta^\vee_{P_2} + \Delta^\vee_{P_3}$ is the 1-cycle given by $\mathscr{E}_*(P_1 \cup P_2 \cup P_3, y_0)$. Therefore, to prove the Proposition, it suffices to show that $\mathscr{M}$ is rationally connected.

Let $\pi : \mathscr{M} \to Y$ be the rational map that sends a cubic surface (S, y_0) that is a cone onto its vertex y_0. The fiber of π over y_0 parametrizes the cubic surfaces in Y that are

cones with vertex y_0. Then S is contained in the singular hyperplane section $Y_{y_0} := Y \cap \bar{T}_{Y,y_0}$ where $\bar{T}_{Y,y_0}$ is the projective tangent space of Y at the point y_0. We may take y_0 a general point so that the cubic hypersurface Y_{y_0} of dimension 7 has an ordinary double point at y_0, and does not contain any $\mathbb{P}^3$, and so that no 3-dimensional linear subspaces in $\mathbb{P}^9$ passing through y_0 is contained in Y. The last condition is satisfied since the 3-dimensional linear subspaces contained in Y form a divisor in Y, as can be shown by a simple dimension counting argument.

Lemma 4.3.9. *Under the assumptions above on y_0, the fiber of π over y_0 is in bijection with the set of planes contained in the Hessian quadric Q_{y_0} of Y_{y_0} at the point y_0.*

Proof. A cubic surface $S = \mathbb{P}^3 \cap Y_{y_0}$ in Y_{y_0} passing through y_0 is a cone with vertex y_0 if and only if the equation f_{y_0} defining Y_{y_0} has vanishing Hessian, which means that the Hessian quadric vanishes on the tangent space $T_{\mathbb{P}^3,y_0}$. □

One knows that the variety of planes in a quadric sixfold is a connected Fano manifold. Thus, by Lemma 4.3.9, the map π is a fibration whose base and general fiber are rationally connected. Therefore, the total space $\mathscr{M}$ is rationally connected [GHS03], as desired. □

Definition 4.3.10. Let $\Delta^\vee$ be an element of $CH_1(F_1(Y))_\mathbb{Q}$ defined by $\frac{1}{3}(\Delta^\vee_{P_1} + \Delta^\vee_{P_2} + \Delta^\vee_{P_3})$ where P_1, P_2, P_3 are the planes as in Proposition 4.3.7.

Corollary 4.3.11. *Let $L \subset Y$ be a 3-dimensional linear space in Y. For any plane $P \subset L$, we have $\Delta^\vee_P = \Delta^\vee$ in $CH_1(F_1(Y))_\mathbb{Q}$.*

Proof. This is because the triple (P,P,P) satisfies Proposition 4.3.7 and thus $3\Delta^\vee_P = 3\Delta^\vee$, from which we conclude. □

Corollary 4.3.12. *Let $L \subset Y$ be a 3-dimensional linear space in Y. Then the image of the natural morphism*

$$CH_1(F_1(L))_\mathbb{Q} \to CH_1(F_1(Y))_\mathbb{Q}$$

is of dimension 1 and is generated by $\Delta^\vee$. In particular, the image does not depend on the choice of $L \in F_3(Y)$.

Remark 4.3.13. We will see, in Proposition 4.3.25, that for any $P \subset Y$, we have $\Delta^\vee_P = \Delta^\vee$.

The geometry of $F_1(Y)$ and its 1-cycles

The following Lemma is proved by an easy dimension count.

Lemma 4.3.14. *Assume Y is general. The linear subspaces $\mathbb{P}^3 \subset Y$ cover a divisor D in Y.*

Corollary 4.3.15. *A general line $\Delta \subset Y$ meets finitely many $\mathbb{P}^3 \subset Y$.*

Indeed, if $\Delta \not\subset D$, the $\mathbb{P}^3 \subset Y$ intersecting Δ are in bijection with the intersection points of Δ and D. Let $F_{3,1} \subset F_3(Y) \times F_1(Y)$ be the set of pairs (x,s) such that $P_x \cap \Delta_s \neq \emptyset$. Let $p : F_{3,1} \to F_3(Y)$ (resp. $q : F_{3,1} \to F_1(Y)$) be the first (resp. second) projection. The second projection $q : F_{3,1} \to F_1(Y)$ is dominant by Lemma 4.3.14 and generically finite by Corollary 4.3.15.

Lemma 4.3.16. *(i) The morphism $q^* : CH_1(F_1(Y)) \to CH_1(F_{3,1})$ is injective.*
(ii) The morphism $p_* \circ q^* : CH_1(F_1(Y))_{hom} \to CH_1(F_3(Y))_{hom}$ is zero.

Proof. (i) follows from the fact that q is dominant. For (ii), we observe that there is a natural birational map

$$\pi : \mathscr{P}_3 \times_Y \mathscr{P}_1 \to F_{3,1},$$

where

$$\mathscr{P}_3 \subset F_3(X) \times Y,\ \mathscr{P}_1 \subset F_1(X) \times Y$$

are the incidence correspondences. The morphism π is commutative with the projections to $F_3(Y)$ and to $F_1(Y)$. By direct calculations,

$$\begin{aligned}\mathscr{P}_3^* \circ \mathscr{P}_{1*} &= p_* \circ \pi_* \circ \pi^* \circ q^* \\ &= p_* \circ q^*\end{aligned}$$

Thus the morphism $p_* \circ q^*$ factors through the morphism $(\mathscr{P}_1)_* : CH_1(F_1(Y))_{hom} \to CH_2(Y)_{hom}$, which is zero because $CH_2(Y)_{hom} = 0$ by [Otw99]. □

The variety $F_{3,1}$ admits a rational map f to the projective bundle $\mathscr{P}_5$ over $F_3(Y)$ whose fiber over $x \in F_3(Y)$ is the set of $\mathbb{P}^4$ containing P_x. The variety $\mathscr{P}_5$ is of dimension 9 since it is a $\mathbb{P}^5$-bundle over a 4-fold $F_3(Y)$. To a pair (P_x, Δ_t) of a $\mathbb{P}^3$ and a line which intersect, this map associates $\langle P_x, \Delta_t \rangle$. We introduce now a desingularization $\tau : \widetilde{F_{3,1}} \to F_{3,1}$ on which f desingularizes to a morphism $\tilde{f} : \widetilde{F_{3,1}} \to \mathscr{P}_5$. We observe now the following: for each $x \in F_3(Y)$ and 4-dimensional space P_x' containing P_x, the intersection $P_x' \cap Y$ is the union $P_x \cup Q_x$ where Q_x is a 3-dimensional quadric intersecting P_x along a 2-dimensional quadric. The general fiber of $\tilde{f}$ over (x, P_x') is birational to the family of lines in the 3-dimensional quadric Q_x. We are now going to prove

Proposition 4.3.17. *Let $z \in CH_1(\widetilde{F_{3,1}})_{\mathbb{Q}}$ be a 1-cycle such that $\tilde{f}_*(z) = 0 \in CH_1(\mathscr{P}_5)$. Then $\tilde{q}_*(z) = \alpha\Delta^\vee \in CH_1(F_1(Y))_{\mathbb{Q}}$ for some $\alpha \in \mathbb{Q}$.*

We will use for this the following general result of Bloch-Srinivas type [Voi14]:

Lemma 4.3.18. *Let $f : Z \to B$ be a surjective projective morphism between algebraic varieties. Let B^0 be an open dense subset of B such that*
(a) $Z - f^{-1}(B^0)$ is of codimension at least 2, and that
(b) Every fiber of f over B^0 has trivial CH_0, i.e., $CH_{0,hom} = 0$.
Let $z \in CH_1(Z)_{\mathbb{Q}}$ be a 1-cycle in Z such that $f_*(z) = 0 \in CH_1(B)_{\mathbb{Q}}$. Then z is supported on the fibers of f over B^0. More precisely, there are points $b_1, \ldots, b_r \in B^0$, such that z is $\mathbb{Q}$-rationally equivalent to a 1-cycle supported on $f^{-1}(b_1) \cup \ldots \cup f^{-1}(b_r)$.

Proof. The case when $\dim B = 1$ is rather trivial, so let us assume $\dim B \geq 2$. Let $W \subset Z$ be a multisection of degree N of $f : Z \to B$. Let $f_W : W \to B$ be the restriction of f to W. Since $Z - f^{-1}(B^0)$ is of codimension at least 2, by Chow moving lemma, we may assume that z is supported on $f^{-1}(B^0)$. We may also assume that none of the components of z lies entirely in the branched locus of f_W. Write $z = z_1 + z_2 + \ldots + z_s - z_{s+1} - \ldots - z_t$ with z_i irreducible curves in $f^{-1}(B^0)$. Consider $B_{z_i} = f(z_i) \subset B^0$. We may assume B_{z_i} is dimension 1 since otherwise the component z_i is supported on the fibers of f. Let $f_i : f^{-1}(B_{z_i}) \to B_{z_i}$ be the restriction of f. Since $\dim B \geq 2$, we may further assume, by Chow moving lemma, that the map $f_i : f^{-1}(B_{z_i}) \to B_{z_i}$ restricted to z_i is a birational map. Since every fiber of f over B^0 has trivial CH_0, the 1-cycle $z_i - \frac{1}{N} f_W^{-1}(B_{z_i})$ in $CH_1(f^{-1}(B_{z_i}))$ restricts to 0 for the general fiber of $f_i : f^{-1}(B_{z_i}) \to B_{z_i}$, so by the Bloch-Srinivas construction [Voi03], the cycle $z_i - \frac{1}{N} f_W^{-1}(B_{z_i})$ is supported on the fibers of f_i. Summing up the components, we find

that the 1-cycle $z - \frac{1}{N} f_W^* f_*(z) \in CH_1(Z)_{\mathbb{Q}}$ is supported on the fibers of f over $\bigcup_i B_{z_i} \subset B^0$. However, $f_*(z) = 0$ by assumption. Hence, $z \in CH_1(Z)_{\mathbb{Q}}$ is supported on the fibers of f over B^0. □

We are going to apply Lemma 4.3.18 to $\tilde{f} : \widetilde{F_{3,1}} \to \mathscr{P}_5$ and to the following open set $\mathscr{P}_5^0$ defined as the set parametrizing the pairs (L_3, P_4) such that
(a) either Q is smooth,
(b) or Q is singular at only one point y and L does not contain y.
It is clear that the fibers of $\tilde{f}$ over $\mathscr{P}_5^0$ are CH_0 trivial. Our first step is thus to check assumption (a) in Lemma 4.3.18. We prove by a case by case analysis the following

Lemma 4.3.19. *$\tilde{f}^{-1}(\mathscr{P}_5^0) \subset \widetilde{F_{3,1}}$ has complement of codimension ≥ 2.*

Proof. The complement R of $\mathscr{P}_5^0$ in $\mathscr{P}_5$ is stratified by the following subsets R_4, R_3, R_2, R_1, where R_i parametrizes $(L, P_4) \in R$ such that the residual Q is of rank i.

Analysis of R_4. The stratum R_4 parametrizes $(L, P_4) \in \mathscr{P}_5$ such that Q is singular at one point (equivalently, the rank of Q is 4) and that L contains the singular point of Q. In this case, Q is a cone from its singular point over a smooth quadric surface.

Sublemma 4.3.20. *For any 3-dimensional linear subspace L contained in Y, the set of 4-dimensional subspaces $P_4 \subset \mathbb{P}^9$ containing L such that Q has a single singular point lying on L has codimension at least 2 in $\mathbb{P}^5 = \{$4-dimensional subspaces $P_4 \subset \mathbb{P}^9$ containing $L\}$.*

Proof. Without loss of generality, let us assume $L = \{(x_0 : \ldots : x_3 : 0 : \ldots : 0)\} \subset \mathbb{P}^9$. The fact that $L \subset Y$ implies that the defining equation of Y is of the form

$$f(x_0, \ldots, x_9) = \sum_{i=4}^{9} x_i Q_i(x_0, \ldots, x_9),$$

where Q_i is a quadratic polynomial in the variables $x_0, \ldots, x_9$, for each $i \in \{4, \ldots, 9\}$. Let $\mathbf{a} := (a_0 : a_1 : \ldots : a_5) \in \mathbb{P}^5$. Then $\mathbf{a}$ determines a dimension 4 linear subspace P_4 containing L as

$$P_4 = \{(x_0 : x_1 : x_2 : x_3 : ta_0, ta_1 : \ldots : ta_5) : (x_0 : x_1 : x_2 : x_3 : t) \in \mathbb{P}^4\}.$$

The corresponded residual quadric hypersuface $Q_{\mathbf{a}}$ is thus defined by

$$Q_{\mathbf{a}}(x_0, x_1, x_2, x_3, t) = \sum_{i=4}^{9} a_{i-4} Q_i(x_0, x_1, x_2, x_3, ta_0, ta_1, \ldots, ta_5).$$

We can identify the quadratic form $Q_{\mathbf{a}}(x_0, x_1, x_2, x_3, t)$ with a 5×5 symmetric matrix that we still denote as $Q_{\mathbf{a}}$. Viewed as a function of $\mathbf{a} \in \mathbb{P}^5$, the matrix $Q_{\mathbf{a}}$ is a section of $\mathrm{Sym}^2(\mathscr{O}(1)_{\mathbb{P}^5} \oplus \mathscr{O}_{\mathbb{P}^5}^4) \otimes \mathscr{O}_{\mathbb{P}^5}(1)$, and thus the locus of $\mathbf{a} \in \mathbb{P}^5$ where $Q_{\mathbf{a}}$ degenerates is a degree 7 hypersurface in $\mathbb{P}^5$. On the other hand, the quadratic polynomial of $Q_{\mathbf{a}} \cap L$ is $Q_{\mathbf{a}}(x_0, x_1, x_2, x_3, t = 0)$, which, viewed as a 4×4 symmetric matrix varing with $\mathbf{a}$, is a section of $\mathrm{Sym}^2(\mathscr{O}_{\mathbb{P}^5}^4) \otimes \mathscr{O}_{\mathbb{P}^5}(1)$, so that the degenerate locus of $Q_{\mathbf{a}} \cap L$ is of degree 4. Therefore, there exists a point $\mathbf{a} \in \mathbb{P}^5$ such that $Q_{\mathbf{a}}$ is singular whereas $Q_{\mathbf{a}} \cap L$ is smooth. Therefore, the locus of $\mathbf{a} \in \mathbb{P}^5$ such that $Q_{\mathbf{a}}$ is singular at exactly one point and that this point lies in L is strictly contained, as a closed subset, in the locus where $Q_{\mathbf{a}}$ is singular at only one point. The latter locus is of codimension 1 in $\mathbb{P}^5$ since it is defined by the vanishing of the determinant of the matrix $Q_{\mathbf{a}}$. Hence, the set of 4-dimensional subspaces $P_4 \subset \mathbb{P}^9$ containing L such that Q has a single singular point lying on L has codimension at least 2 in $\mathbb{P}^5$, as desired. □

Sublemma 4.3.21. *The fiber of $\tilde{f}:\widetilde{F_{3,1}}\to\mathscr{P}_5$ over an element (L,P_4) such that the corresponding residual quadric Q has only one singularity is isomorphic to the union*

$$\mathscr{P}^{\vee}_{2,1}\cup\mathscr{P}^{\vee}_{2,2},$$

where $\mathscr{P}^{\vee}_{2,i}$ $(i=1,2)$ is a $\mathbb{P}^2$-bundle over $\mathbb{P}^1$, and $\mathscr{P}^{\vee}_{2,1}\cap\mathscr{P}^{\vee}_{2,2}$ is a smooth quadric surface.

Proof. The singular quadric hypersurface Q is a cone with vertex at its only singular point y over a smooth quadric surface Q'. Each line in Q', together with x determines a plane, and every line in Q lies in some of these planes. There are two pencils of lines in Q', each of which induces a pencil of planes in Q'. Each plane contains a $\mathbb{P}^2$ of lines. Hence, the set of lines in Q is the union of two $\mathbb{P}^2$-bundles over $\mathbb{P}^1$. The intersection of these $\mathbb{P}^2$-bundles is the set of lines in Q that pass through the singular point y, which is in turn isomorphic to Q'. □

Sublemma 4.3.20 implies that $R_4\subset\mathscr{P}_5$ is of codimension at least 2. Sublemma 4.3.21 shows that the dimension of the fiber of $\tilde{f}$ over R_4 is 3, the same with that of the general fiber of $\tilde{f}:\widetilde{F_{3,1}}\to\mathscr{P}_5$. Hence, the codimension of $\tilde{f}^{-1}(R_4)$ is at least 2. This complete the analysis of R_4.

Analysis of R_3. The stratum R_3 parametrizes $(L,P_4)\in\mathscr{P}_5$ such that Q is of rank 3. In this case, Q is singular along a line and is a cone with its vertex one of its singular point over a quadric cone surface.

Sublemma 4.3.22. *The stratum R_3,R_2,R_1 is of codimension $3,6,10$, respectively, in $\mathscr{P}_5$. In particular, $R_1=\emptyset$.*

Proof. Let $\mathscr{E}_4$ and $\mathscr{Q}_6$ be the tautological subbundle, respectively, tautological quotient bundle over $F_3(Y)$. Then $\mathscr{P}_5$ is nothing but the projectivization of $\mathscr{Q}_6$. Let $\mathscr{H}$ be the Hopf bundle of $\mathbb{P}(\mathscr{Q}_6)$, which is a subbundle of $\pi^*\mathscr{Q}_6$. Let $\mathscr{F}_5$ be the kernel of the composite of the canonical maps $V_{10}\to\pi_*\mathscr{Q}_6\to\pi_*\mathscr{Q}_6/\mathscr{H}$, where the first map is the universal quotient map of bundles of $\mathbb{P}(\mathscr{Q}_6)$. Then there is a natural exact sequence of vector bundles on $\mathbb{P}(\mathscr{Q}_6)$:

$$0\to\pi^*\mathscr{E}_4\to\mathscr{F}_5\to\mathscr{H}\to 0.$$

The defining equation of $Y\subset\mathbb{P}^9$ gives a section of the bundle $\mathrm{Sym}^3\mathscr{F}_5^*$ whose image in $\pi^*\mathrm{Sym}^3\mathscr{E}_4^*$ vanishes since the fibers of $\mathbb{P}(\mathscr{E}_4)$ are by definition 3-dimensional linear subspaces in Y. Hence, the universal residual quadric hypersurface is defined by a section of the bundle $\mathrm{Sym}^2\mathscr{F}_5^*\otimes\mathscr{H}^*$. Notice that $\mathrm{Sym}^2\mathscr{F}_5^*\otimes\mathscr{H}^*$ is generated by global sections. Since Y is general, the locus where such a quadratic form is of rank $\leq r$ is $\binom{6-r}{2}$, for $r\in\{0,\dots,4\}$, as desired. □

Sublemma 4.3.23. *The fiber of $\tilde{f}:\widetilde{F_{3,1}}\to\mathscr{P}_5$ over a point (L,P_4) in R_3 is of dimension 3.*

Proof. The residual quadric hypersurface Q is singular along a line and is a cone from one of its singular point over a quadric cone surface. By the same argument as in Sublemma 4.3.21, the variety of lines in Q is isomorphic to a $\mathbb{P}^2$-bundle over a smooth conic, thus has dimension 3. □

By Sublemma 4.3.22 and Sublemma 4.3.23, the codimension of $\tilde{f}^{-1}(R_3)$ is 3. This completes the analysis of R_3.

Analysis of R_2. The stratum R_2 parametrizes $(L,P_4)\in\mathscr{P}_5$ such that Q is of rank 2. In this case, Q is a union of two 3-dimensional linear subspaces intersecting along a plane.

It is clear that in this case, the variety of lines in Q is isomorphic to the union of two $\mathrm{Gr}(2,4)$ intersecting along a $\mathbb{P}^2$. Hence, the dimension of fibers of $\tilde{f}$ over R_2 is 4. By Sublemma 4.3.22, the codimension of $\tilde{f}^{-1}(R_2)$ is 5. This completes the analysis of R_2.

Analysis of R_1. The stratum R_1 parametrizes $(L,P_4) \in \mathscr{P}_5$ such that Q is of rank 1. By Sublemma 4.3.22, this case does not happen.

The case where Q is of rank 0 does not happen, since in this case, P_4 is contained in Y, which cannot happen if Y is a general cubic eightfold. □

Lemma 4.3.24. *For any (L,P_4) in $\mathscr{P}_5^0$, with associated 3-dimensional quadric Q, the natural morphism*

$$CH_1(F_1(L\cap Q))_{\mathbb{Q}} \to CH_1(F_1(Q))_{\mathbb{Q}}$$

is surjective.

Proof. Recall that after the proof of Lemma 4.3.18, we have divided the points in $\mathscr{P}_5^0$ into two cases.

Case (a). In this case, $CH_1(F_1(Q))_{\mathbb{Q}} = \mathbb{Q}$ since $F_1(Q)$ is a connected Fano manifold. Therefore, $CH_1(F_1(L\cap Q))_{\mathbb{Q}} \to CH_1(F_1(Q))_{\mathbb{Q}}$ has to be surjective.

Case (b). In this case, Q is a cone from its singular point y over a smooth quadric surface Q'. By Lemma 4.3.21, the Fano variety of lines $F_1(Q)$ is isomorphic to $\mathscr{P}_{2,1}^\vee \cup \mathscr{P}_{2,2}^\vee$ where $\mathscr{P}_{2,i}^\vee$ is a $\mathbb{P}^2$-bundle over $\mathbb{P}^1$ and the $\mathscr{P}_{2,1}^\vee \cap \mathscr{P}_{2,2}^\vee$ is a smooth quadric surface Q'. The variety $F_1(L\cap Q)$ is a disjoint union of two lines ℓ_1 and ℓ_2, each representing a pencil of lines on the smooth quadric surface $L\cap Q$. Hence, $CH_1(F_1(L\cap Q))$ is generated freely by two cycles z_1 and z_2 where z_i represents ℓ_i for $i=1,2$. Let us analyze $CH_1(F_1(Q))$. Let us consider $\mathscr{P}_{2,1}^\vee$ which is a $\mathbb{P}^2$-bundle over ℓ_1. Let $p_1 : \mathscr{P}_{2,1}^\vee \to \ell_1$ be the corresponding map, and let h be the auti-tautological class of this projective bundle. The Chow group $CH_1(\mathscr{P}_{2,1}^\vee)$ is then generated by $h^2\cdot p_1^*(\ell_1)$ and $h\cdot p_1^*(pt)$. The image of the class $h^2\cdot p_1^*(\ell_1)$ in $CH_1(F_1(Q))$ is the same as the class of z_1 in $CH_1(F_1(Q))$ and the image of the class $h\cdot p_1^*(pt)$ is the as the class of z_2 in $CH_1(F_1(Q))$. Hence, the image of $CH_1(F_1(L\cap Q)) \to CH_1(F_1(Q))$ contains the image of $CH_1(\mathscr{P}_{2,1}^\vee) \to CH_1(F_1(Q))$. Similarly, the image of $CH_1(F_1(L\cap Q)) \to CH_1(F_1(Q))$ contains the image of $CH_1(\mathscr{P}_{2,2}^\vee) \to CH_1(F_1(Q))$. Therefore, $CH_1(F_1(L\cap Q)) \to CH_1(F_1(Q))$ is surjective. □

We finally conclude the proof of Proposition 4.3.17.

Proof of Proposition 4.3.17. Let $z \in CH_1(\widetilde{F_{3,1}})_{\mathbb{Q}}$ be a 1-cycle such that $\tilde{f}_*(z) = 0 \in CH_1(\mathscr{P}_5)_{\mathbb{Q}}$. Lemma 4.3.19 and the discussion above shows that Lemma 4.3.18 can be applied to the map $\tilde{f} : \widetilde{F_{3,1}} \to \mathscr{P}_5$ with $\mathscr{P}_5^0$ the open dense subset, so that we conclude that z is supported on fibers of $\tilde{f}$ over $\mathscr{P}_5^0$. Write $z = z_1 + \ldots + z_r \in CH_1(\widetilde{F_{3,1}})_{\mathbb{Q}}$ where z_i is supported on $F_1(Q_i)$ with Q_i is a residual quadric hypersurface coming from a pair $(L_i, P_{4,i})$ in $\mathscr{P}_5^0$. By Lemma 4.3.24, for each i, the the natural morphism $CH_1(F_1(Q_i\cap L_i))_{\mathbb{Q}} \to CH_1(F_1(Q_i))_{\mathbb{Q}}$ is surjective. Hence, $\tilde{q}_*(z)$ lies in the sum of the images of $CH_1(F_1(L_i))_{\mathbb{Q}} \to CH_1(F_1(Y))_{\mathbb{Q}}$. By Corollary 4.3.12, the cycle $\tilde{q}_*(z)$ is a multiple of $\Delta^\vee$, as desired. □

We next prove that all $\Delta_P^\vee$ has the same Chow class in $CH_1(F_1(Y))_{\mathbb{Q}}$.

Proposition 4.3.25. *Let $P \subset Y$ be a plane. Then $\Delta_P^\vee = \Delta^\vee$ in $CH_1(F_1(Y))_{\mathbb{Q}}$.*

Proof. Since the rational equivalence class is a countable union of closed algebraic subsets [Voi14, Section 1.1.1], we may assume $P \subset Y$ is general. Let $P' \subset Y$ be another plane, general among all planes in Y intersecting P along a line l. By Corollary 4.3.15, there is a

3-dimensional linear space $L \subset Y$ such that l intersects L at a point y. As P and P' are general, we may assume that both P and P' intersect L at only one point y. The line of lines in P passing through y naturally lifts to a curve $Z_1 \subset \widetilde{F_{3,1}}$ contained in the fiber of $\pi \circ \tilde{f}$ over the point l_3 of $F_3(Y)$ parameterizing L_3. We consider the 1-cycle $z := [Z_1] - [Z_1'] \in CH_1(\widetilde{F_{3,1}})_{\mathbb{Q}}$. It is clear that $\tilde{f}_*(z) = 0$ in $CH_1(\mathscr{P}_5)_{\mathbb{Q}}$ since $\tilde{f}_*([Z_1])$ and $\tilde{f}_*([Z_1'])$ is represented by two lines in the fiber of the $\mathbb{P}^5$-bundle $\mathscr{P}_5$ over the point $L \in F_3(Y)$. By Proposition 4.3.17, we have $\tilde{q}_*(z) = \alpha\Delta^\vee$ in $CH_1(F_1(Y))_{\mathbb{Q}}$. But it is clear that $\tilde{q}_*([Z_1]) = \Delta_P^\vee$ and $\tilde{q}_*([Z_1']) = \Delta_{P'}^\vee$, and that $\alpha = 0$ by degree reasons. Hence, $\Delta_P^\vee = \Delta_{P'}^\vee$. There exists a $P_3 \cong \mathbb{P}^3$ containing P and P' and the intersection $P_3 \cap Y$ is the union of three planes P, P', P''. The same argument as above shows that $\Delta_P^\vee = \Delta_{P'}^\vee = \Delta_{P''}^\vee$. Finally, Proposition 4.3.7 shows that $\Delta_P^\vee = \Delta^\vee$, as desired. □

We now conclude the proof of Theorem 4.3.6 in this section.

Proof of Theorem 4.3.6. It is not hard to show that $H^2(F_1(Y), \mathbb{Q}) = \mathbb{Q}$ since the restriction map $H^2(\mathrm{Gr}(2,10), \mathbb{Q}) \to H^2(F_1(Y), \mathbb{Q})$ is an isomorphism [DM98]. It suffices to prove that $CH_1(F_1(Y))_{\mathbb{Q},hom} = 0$. Let $\alpha \in CH_1(F_1(Y))_{\mathbb{Q},hom}$ and let $z = \tilde{q}^*\alpha \in CH_1(\widetilde{F_{3,1}})_{\mathbb{Q},hom}$. Since $\tilde{q}_*(z) = \deg\tilde{q} \cdot \alpha$, it suffices to prove that $\tilde{q}_*(z) = 0$.

Since $\mathscr{P}_5$ is a $\mathbb{P}^5$-bundle over $F_3(Y)$, we have $CH_1(\mathscr{P}_5) = h^5 \cdot \pi^* CH_1(F_3(Y)) \oplus h^4 \cdot \pi^* CH_0(F_3(Y))$. Lemma 4.3.16 shows that $\tilde{p}_* z = 0 \in CH_1(F_3(Y))_{\mathbb{Q},hom}$. Hence, $\tilde{f}_*(z) \in h^4 \cdot \pi^* CH_0(F_3(Y))_{\mathbb{Q},hom}$. Write $\tilde{f}_*(z) = w_1 + \ldots + w_s$ where w_i is a 1-cycle supported on the fiber of π over a point $L_i \in F_3(Y)$. Let P_i be a plane in Y that intersects with L_i at only one point y_i. Let $P_{5,i}$ be the 5-dimensional linear subspace spanned by L_i and P_i. Let z_i be the class in $CH_1(\widetilde{F_{3,1}})$ represented by the variety $Z_i := \{(L_i, l) : y_i \in l \subset P_i\}$. Since the fibers of $\pi : \mathscr{P}_5 \to F_3(Y)$ are projective spaces $\mathbb{P}^5$, the cycle w_i is proportional to the class represented by the variety $\{(L_i, P_4) \in \mathscr{P}_5 : P_4 \subset P_{5,i}\}$, which is the image of Z_i under $\tilde{f}$. Hence, with an appropriate choice of coefficients $a_i \in \mathbb{Q}$, we have

$$\tilde{f}_*(z) = \sum_i w_i = \sum_i a_i \tilde{f}_* z_i = \tilde{f}_*(\sum_i a_i z_i).$$

By Proposition 4.3.17, we conclude that

$$\tilde{q}_*(z) = \tilde{q}_*(\sum_i a_i z_i) = \sum_i a_i \Delta_{P_i}^\vee = a\Delta^\vee,$$

where $a = \sum_i a_i$. The last equality is due to Proposition 4.3.25. Since $\tilde{q}_*(z)$ is homologue to 0, the coefficient $a = 0$. Hence, $\tilde{q}_*(z) = 0$, as desired. This terminates the proof of Theorem 4.3.6. □

4.3.3 Proof of Theorem B

We prove in this section Theorem B from the introduction.

Theorem B. *Let $Y \subset \mathbb{P}^9$ be a general cubic 8-fold, and let $X = F_2(Y)$ be the Fano variety of planes in Y. Let $\Psi : X \dashrightarrow X$ be the Voisin map. Then for any $z \in CH_0(X)_{hom}$:*

$$\Psi_* z = -8z.$$

Putting together Theorem 4.3.1, Proposition 4.3.3, Proposition 4.3.4 and Theorem 4.3.6, we conclude that formula (4.11) becomes

$$\Psi_* z = -8z + \gamma' I_{2*} z \tag{4.16}$$

for any cycle γ' of the form $ac_1^3 + b'c_1c_2 + c'c_3$ on X, where the number a is determined by the class γ of (4.11) by $\gamma = ac_1^3 + bc_1c_2 + cc_3$, and b', c' are arbitrary. We take for γ' a multiple of the class of the fixed locus of F of Ψ. Indeed, Proposition 4.3.39 proved in Section 4.3.3 says that the class of F in $\mathrm{CH}^3(X)$ has a nonzero coefficient in c_1^3. Theorem B then follows form

Theorem 4.3.26. *The fixed locus F is a constant cycle subvariety in X.*

Indeed, Theorem 4.3.26 says that the natural morphism $CH_0(F)_{hom} \to CH_0(X)_{hom}$ is zero.

Remark 4.3.27. Theorem 4.3.26 had been proved in [Voi08] in the case $r = 1$.

Proof of Theorem 4.3.26

Let $\mathscr{P}_{2,1} := \{(x,l) \in X \times F_1(Y) : l \subset P_x\}$ be the incidence variety.

Remark 4.3.28. We have $I_2 =^t \mathscr{P}_{2,1} \circ \mathscr{P}_{2,1}$ as self-correspondence of X.

Let $x \in X = F_2(Y)$ be a general point and let $x' = \Psi(x)$ where $\Psi : X \dashrightarrow X$ is the Voisin map. In what follows, we use the following notation: for a plane P, the dual of P, defined as the set of lines in P, is denoted by $P^\vee$.

Proposition 4.3.29. *The cycle $P_{x'}^\vee - 4P_x^\vee \in CH_2(F_1(Y))_\mathbb{Q}$ does not depend on the choice of $x \in X$.*

Admitting Proposition 4.3.29, we conclude the proof of Theorem 4.3.26.

Proof of Theorem 4.3.26. By Remark 4.3.28, we have

$$\mathscr{P}_{2,1}^* \circ \mathscr{P}_{2,1*} = I_{2*} : CH_0(X) \to CH_3(X).$$

If $x \in F \subset X$, then in the statement of Proposition 4.3.29, $x' = x$ and thus $P_x^\vee \in CH_2(F_1(Y))_\mathbb{Q}$ is independent of the choice of $x \in F$. Hence, for any $x_1, x_2 \in F$, we have $\mathscr{P}_{2,1*}(x_1 - x_2) = P_{x_1}^\vee - P_{x_2}^\vee = 0 \in CH_2(F_1(Y))_\mathbb{Q}$. Therefore, $I_{2*}(x_1 - x_2) = \mathscr{P}_{2,1}^* \mathscr{P}_{2,1*}(x_1 - x_2) = 0 \in CH_0(X)_\mathbb{Q}$. Now, if in Equation (4.11)

$$\Psi_* z = -8z + \gamma \cdot I_{2*}(z),$$

we take $z = x_1 - x_2$, we get $z = -8z \in CH_0(X)_\mathbb{Q}$. Therefore, $z = 0 \in CH_0(X)$ as $CH_0(X)$ is torsion-free. This implies $x_1 = x_2 \in CH_0(X)$. Since $x_1, x_2 \in F$ are arbitrarily chosen, we conclude that F is a constant cycle subvariety. □

The rest of this section is devoted to the proof of Proposition 4.3.29. The proof relies on the geometry of cycles in a cubic fourfold Y_4 and its variety of lines $F_1(Y_4)$, which has been studied in [Voi08] and [SV15]. the following relation is established in [Voi08] (see also [SV15]).

Theorem 4.3.30 (Voisin [Voi08])**.** *For a cubic fourfold $Y_4 \subset \mathbb{P}^5$ containing a plane P, let $P^\vee \subset F_1(Y_4)$ denote the variety of lines within P, and let $D_P \subset F_1(Y_4)$ represent the divisor comprising lines in Y_4 intersecting P. With $l \subset CH^1(F_1(Y_4))$ being the restriction of the Plücker line bundle class from $\mathrm{Gr}(2,6)$, there exist constants $\alpha, \beta \in \mathbb{Q}$, and $\gamma \in CH^2(F_1(Y_4))_\mathbb{Q}$, where $\alpha \neq 0$ and γ is a restriction of a class $\delta \in CH^2(\mathrm{Gr}(2,6))_\mathbb{Q}$ that is independent of the chose of the plane P and the cubic fourfold Y_4, such that:*

$$P^\vee = \alpha D_P^2 + \beta D_P \cdot l + \gamma$$

within $CH^2(F_1(Y_4))_\mathbb{Q}$.

We will partially generalize this relation to the case where Y_4 has mild singularities.

Corollary 4.3.31. *Consider a cubic hypersurface $Y_4 \subset \mathbb{P}^5$ with at most simple double points as singularities, containing a plane P. Let $F_1(Y_4)_{sm}$ denote the smooth part of $F_1(Y_4)$. Define $P^\vee \subset F_1(Y_4)_{sm}$, $D_P \subset F_1(Y_4)_{sm}$, and $l \subset CH^1(F_1(Y_4)_{sm})$ as in Theorem 4.3.30, but restricted to the smooth part of $F_1(Y_4)$. There exist constants $\alpha, \beta \in \mathbb{Q}$, and $\gamma \in CH^2(F_1(Y_4)_{sm})_{\mathbb{Q}}$, with $\alpha \neq 0$ and γ as a restriction of a class $\delta \in CH^2(\mathrm{Gr}(2,6))_{\mathbb{Q}}$ that is independent of the chose of the plane P and the cubic fourfold Y_4, such that:*

$$P^\vee = \alpha D_P^2 + \beta D_P \cdot l + \gamma \tag{4.17}$$

in $CH^2(F_1(Y_4)_{sm})_{\mathbb{Q}}$.

Proof. Let us consider the construction of the family $\mathscr{F}$ over B, where B parametrizes pairs (P, f) with P being a plane in $\mathbb{P}^5$ and f a cubic polynomial such that the hypersurface Y_f defined by f has at most simple double points as singularities, together with the condition that $f_{|P} = 0$. Let

$$\mathscr{F} = \{((P,f),l) \in B \times \mathrm{Gr}(2,6) : f|_l = 0\}\}.$$

In such a way, we make the Fano variety of lines $F_1(Y_f)$ into family over B.

Lemma 4.3.32. *The family $p : \mathscr{F} \to B$ is flat.*

Proof. Each fiber of p is a subvariety that is defined as the zero locus of the vector bundle $\mathrm{Sym}^3 \mathscr{E}^*$ on $\mathrm{Gr}(2,6)$, with the expected dimension, where $\mathscr{E}$ is the tautological subbundle of $\mathrm{Gr}(2,6)$. By Koszul's resolution, each fiber has the same Hilbert polynomial. This implies that the family is flat. □

Within the family $\mathscr{F}$, we have the following subvarieties. $\mathscr{P}^\vee := \{((P,f),l) \in B \times \mathrm{Gr}(2,6) : l \subset P\}$ and $\mathscr{D} := \{((P,f),l) \in \mathscr{F} : l \cap P \neq \emptyset\}$, representing lines within P and lines intersecting P, respectively. Additionally, let $q : \mathscr{F} \to \mathrm{Gr}(2,6)$ be the second projection. Let $\mathscr{L} = q^* \mathscr{O}_{\mathrm{Gr}(2,6)}(1)$ be the pull-back of the Plücker line bundle. Let $\Gamma = q^* \delta \in CH^2(\mathscr{F})_{\mathbb{Q}}$, where $\delta \in CH^2(\mathrm{Gr}(2,6))_{\mathbb{Q}}$ be the constant class as defined in Theorem 4.3.30. Let $\alpha, \beta \in \mathbb{Q}$ be as in Theorem 4.3.30. We consider the algebraic cycle $\mathscr{Z} = \mathscr{P}^* - \alpha \mathscr{D}^2 - \beta \mathscr{D} \cdot \mathscr{L} - \Gamma$. Theorem 4.3.30 implies that $\mathscr{Z}|_{\mathscr{F}_t} = 0 \in CH^2(\mathscr{F}_t)_{\mathbb{Q}}$ for $t \in B$ with smooth fibers. By the specialization of algebraic cycles [Voi15, Proposition 1.4], we conclude that $\mathscr{Z}|_{\mathscr{F}_t} = 0 \in CH^2(\mathscr{F}_t)$ for all $t \in B$. For a singular fiber $\mathscr{F}_t$, we can restrict further to the smooth part of $\mathscr{F}_t$ and we get the desired result. □

Remark 4.3.33. The reason we do not achieve the relation (4.17) for the whole of $F_1(Y_4)$ is that the divisor D_P might encompass the singular locus of $F_1(Y_4)$, rendering it not a Cartier divisor, hence D_P^2 is not well-defined. However, upon restriction to the smooth part, all components are well-defined, and the restriction of $\mathscr{D}^2$ equates to D_P^2.

We will also need the following

Lemma 4.3.34. *Let $Y \subset \mathbb{P}^9$ be a general cubic eightfold and let $P \subset Y$ be a general plane contained in Y. Let H_5 be a general linear subspace of $\mathbb{P}^9$ containing P such that $H_5 \cap Y =: Y_4$ is a cubic hypersurface containing the plane P. Let $j : F_1(Y_4) \hookrightarrow F_1(Y)$ be the natural inclusion map. Let $D_P \subset F_1(Y_4)$ be as in Theorem 4.3.30. Then the class $j_* D_P \in CH_3(F_1(Y))$ is independent of the choice of the plane $P \subset Y$ and of the linear subspace $H_5 \subset \mathbb{P}^9$.*

Proof. Let $q : \mathscr{P}_{1,0} \to Y$ and $p : \mathscr{P}_{1,0} \to F_1(Y)$ be the universal correspondence of Y and $F_1(Y)$. Let $\Sigma_P \subset F_1(Y)$ be the variety of lines in Y that intersects the plane P. The class $\Sigma_P \in CH^5(F_1(Y))$ does not depend on the choice of $P \subset Y$, since $\Sigma_P = \mathscr{P}^*_{1,0}(P)$ and since $CH_2(Y) = \mathbb{Z}P$ by [Otw99]. Let $\tilde{\Sigma}_P \subset \mathscr{P}_{1,0}$ be the preimage of $P \subset Y$ via q. Then similarly, the Chow class of $\tilde{\Sigma}_P$ in $\mathscr{P}_{1,0}$ does not depend on the choice of P. Let us define two vector bundles $\mathscr{E}$ and $\mathscr{H}$ on $\mathscr{P}_{1,0}$ as follows. $\mathscr{E}$ is the pull-back of the universal subbundle over $F_1(Y) \subset \mathrm{Gr}(2,10)$ via p and $\mathscr{H}$ is the pull-back of the Hopf bundle over $Y \subset \mathbb{P}^9$ via q. On $\tilde{\Sigma}_P$, we have a natural inclusion map $\mathscr{H}|_{\tilde{\Sigma}_P} \hookrightarrow \mathscr{E}|_{\tilde{\Sigma}_P}$ that induces a sujective morphism of vector bundles $\phi : \mathscr{E}|^*_{\tilde{\Sigma}_P} \to \mathscr{H}|^*_{\tilde{\Sigma}_P}$ that fits into the short exact sequence

$$0 \to (\det \mathscr{E}^* \otimes \mathscr{H})|_{\tilde{\Sigma}_P} \to \mathscr{E}|^*_{\tilde{\Sigma}_P} \to \mathscr{H}|^*_{\tilde{\Sigma}_P} \to 0. \tag{4.18}$$

Over $\tilde{\Sigma}_P$, the defining equations of $H_5 \subset \mathbb{P}^9$ induces a section s of $\mathscr{E}^*|^{\oplus 4}_{\tilde{\Sigma}_P}$ that is zero when projected to $H^0(\tilde{\Sigma}_P, \mathscr{H}^*|^{\oplus 4}_{\tilde{\Sigma}_P})$. Hence, we can view s as a section σ_s of $(\det \mathscr{E}^* \otimes \mathscr{H})|^{\oplus 4}_{\tilde{\Sigma}_P}$. Let $\tilde{D}_P$ be the zero locus of σ_s. Then $\tilde{D}_P$ parametrizes the pairs $(l,y) \in F_1(Y) \times Y$ such that $l \subset H_5$ and $l \cap P = y$. By the projection formula, we find that the Chow class of $\tilde{D}_P$ in $\mathscr{P}_{1,0}$ is $\tilde{\Sigma}_P \cdot c_1((\det \mathscr{E}^* \otimes \mathscr{H}))^4$, which is independent of the choice of P and H_5. Since $D_P = p_* \tilde{D}_P$, the Chow class of D_P in $F_1(Y)$ is independent of the choice of P and H_5 as well. □

For a plane $P \subset Y$, let $P^\vee \subset F_1(Y)$ be the subvariety of lines contained in P. Let $l \in CH^1(F_1(Y))$ be the restriction of the Plücker line bundle class of $\mathrm{Gr}(2,10)$. Let $Y_4 \subset Y$ be a linear section of Y that has at most simple double points as singularities. Let $\Sigma \subset F_1(Y_4)$ be the singular locus of $F_1(Y_4)$ and let $j : F_1(Y_4)_{sm} \hookrightarrow F_1(Y) - \Sigma =: F_1(Y)^0$ be the inclusion map. Let D_P be defined as in Corollary 4.3.31. Then Corollary 4.3.31 and Lemma 4.3.34 imply the following

Corollary 4.3.35. *In $CH_2(F_1(Y)^0)_{\mathbb{Q}}$, we have the following relation*

$$P^\vee = \alpha j_*(D_P^2) + c|_{F_1(Y)^0},$$

where $\alpha \neq 0$ is a rational number and $c \in CH_2(F_1(Y))_{\mathbb{Q}}$ is a Chow class that is independent of the choice of the plane P.

Proof of Proposition 4.3.29. By the definition of the Voisin map, there is a unique linear subspace $H \subset \mathbb{P}^9$ of dimension 3, such that $H \cap Y = 2P_x + P_{x'}$. Let $H_5 \subset \mathbb{P}^9$ be a linear subspace containing H (thus containing both P_x and $P_{x'}$), and let $Y_4 = Y \cap H_5$.

Lemma 4.3.36. *For a general choice of H_5, the cubic hypersurface Y_4 has 4 simple double points as the only singularities.*

Proof. The base locus of the linear system $L = \{H_5' \cap Y : H \subset H_5' \subset \mathbb{P}^9\}$ is $P_x \cup P_{x'}$. Hence, by Bertini's theorem, for a general H_5, the cubic hypersurface $Y_4 := Y \cap H_5$ is smooth outside $P_x \cup P_{x'}$. Let us write $H_5 = \{(x_0 : x_1 : \ldots : x_5)\}$ and we can assume that $H = \{(x_0 : x_1 : x_2 : x_3 : 0 : 0)\}$, $P_x = \{(x_0 : x_1 : x_2 : 0 : 0 : 0)\}$ and $P_{x'} = \{(0 : x_1 : x_2 : x_3 : 0 : 0)\}$. The fact that $H \cap Y_4 = 2P_x + P_{x'}$ means that the defining equation of $Y_4 \subset H_5$ can be written as

$$f(x_0, \ldots, x_5) = x_3^2 x_0 + x_4 Q_1(x_0, \ldots, x_5) + x_5 Q_2(x_0, \ldots, x_5),$$

where Q_1 and Q_2 are quadratic polynomials. Let $y \in Y_4$ be a singular point. Since $y \in H$, one must have $x_4(y) = x_5(y) = 0$. Writing $f_{x_i} = \frac{\partial f}{\partial x_i}$, we find by direct calculations that $f_{x_0}(y) =$

$x_3(y)^2$, $f_{x_1}(y) = f_{x_2}(y) = 0$, $f_{x_3}(y) = 2x_0(y)x_3(y)$, $f_{x_4}(y) = Q_1(y)$ and $f_{x_5}(y) = Q_2(y)$. The fact $y \in Y_4$ is singular implies that $f_{x_i}(y) = 0$ for any x_i. Taken together, we find that $y \in H_5$ satisfies the following system of equations

$$\begin{cases} x_4 & = 0 \\ x_5 & = 0 \\ x_3 & = 0 \\ Q_1(x_0,\ldots,x_5) & = 0 \\ Q_2(x_0,\ldots,x_5) & = 0 \end{cases}$$

By the generality of $Y \subset \mathbb{P}^9$ and H_5, the solutions of this system of equations are four points in H_5. To show that the four singular points are simple double points. We do a local check. Let y be one of the singular points. Up to a change of coordinates of P_x, we may assume without loss of generality that $x_0(y) \neq 0$ and $x_1(y) = x_2(y) = 0$, namely, $y = (1:0:0:0:0:0) \in H_5$. On the open affine subset U_0 of H_5 defined by $x_0 = 1$, the cubic hypersurface $Y_4 \cap U_0$ is defined by the equation $x_3^2 + x_4Q_1(1,x_1,\ldots,x_5) + x_5Q_2(1,x_1,\ldots,x_5)$. The Taylor expansion of this polynomial around the point $y = (0,0,0,0,0) \in \mathbb{A}^5 \cong U_0$ is the polynomial itself. The fact that this polynomial does not have degree 1 term is simply because y is a singular point. To show that y is a double point, we only need to make sure that the degree 2 term of the expression $x_3^2 + x_4Q_1(1,x_1,\ldots,x_5) + x_5Q_2(1,x_1,\ldots,x_5)$ is non-degenerate, and this condition is clearly satisfied for general Q_1 and Q_2. □

Let y_1, y_2, y_3, y_4 be the four singular points of Y_4. Let $\Sigma \subset F_1(Y_4)$ be the singular locus of $F_1(Y_4)$. Then Σ is the union of four surfaces Σ_1, Σ_2, Σ_3 and Σ_4, parametrizing the lines in Y_4 that pass through the point y_1, y_2, y_3 and y_4, respectively. Let $F_1(Y)^0 = F_1(Y) - \Sigma$ Let $j : F_1(Y_4)_{sm} \hookrightarrow F_1(Y)^0$ be the natural inclusion. By Corollary 4.3.35, we have

$$P_x^* = \alpha j_*(D_{P_x}^2) + c'|_{F_1(Y)^0} \in CH_2(F_1(Y))_{\mathbb{Q}} \tag{4.19}$$

and

$$P_{x'}^* = \alpha j_*(D_{P_{x'}}^2) + c'|_{F_1(Y)^0} \in CH_2(F_1(Y)^0)_{\mathbb{Q}} \tag{4.20}$$

for some constant $c' \in CH_2(F_1(Y))_{\mathbb{Q}}$. Now let $\mathscr{P} = \{(l,y) \in F_1(Y_4)_{sm} \times Y_4 : y \in l\}$ be the incidence correspondence, then for any plane $P \subset Y_4$, we have $D_P = \mathscr{P}^*(P)$. It is clear that $2P_x + P_{x'} = h^2$ in $CH_2(Y_4)$, so $2D_{P_x} + D_{P_{x'}} = \mathscr{P}^*(h^2) = l \in CH_3(F_1(Y_4)_{sm})$. Taken into account of the equations (4.19) and (4.20), we find that

$$P_{x'}^* = 4P_x^* - 4\alpha j_* D_{P_x} \cdot l + \alpha l^2 - 3c' \in CH_2(F_1(Y)^0)_{\mathbb{Q}}.$$

Taking into account of Lemma 4.3.34, $P_{x'}^* - 4P_x^* = c|_{F_1(Y)^0} \in CH_2(F_1(Y)^0)_{\mathbb{Q}}$, where $c \in CH_2(F_1(Y))_{\mathbb{Q}}$ is a constant 2-cycle on $F_1(Y)$. Hence, in $F_1(Y)$, the cycle $P_{x'}^* - 4P_x^* - c$ is supported on Σ, the singular locus of $F_1(Y_4)$. Thus, we need to understand the geometry and Chow classes of Σ.

Following Lemma 4.3.36, let y_1, y_2, y_3, y_4 be the four singular points of Y_4. Then Σ is the union of four surfaces Σ_1, Σ_2, Σ_3 and Σ_4, parametrizing the lines in Y_4 that pass through the point y_1, y_2, y_3 and y_4, respectively. We have the following two lemmas about the geometry of the surfaces Σ_i that we will prove later.

Lemma 4.3.37. *The surfaces Σ_1, Σ_2, Σ_3 and Σ_4 are irreducible. In particular, $CH_2(\Sigma) = \oplus_{i=1}^4 CH_2(\Sigma_i)$.*

Lemma 4.3.38. *Let $H_5 \subset \mathbb{P}^9$ be a general linear subspace of dimension 5 such that the linear section $Y_4 := Y \cap H_5$ has only simple double singularities and let $y \in Y_4$ be a singular point. Let S be the surface of lines in Y_4 that pass through the point y. Then the Chow class of S in $F_1(Y)$ does not depend on the choice of H_5 and y.*

By Lemma 4.3.37 and the fact that $P_{x'} - 4P_x - c$ is supported on Σ, we conclude that $P_{x'} - 4P_x - c = \sum_{i=1}^4 a_i\Sigma_i$, with $a_i \in \mathbb{Q}$. The cohomological class of $P_{x'} - 4P_x - c$ is clearly a constant, thus the cohomological class of $\sum_{i=1}^4 a_i\Sigma_i$ is constant. Lemma 4.3.38 then implies that the Chow class of $\sum_{i=1}^4 a_i\Sigma_i$ is also constant. Therefore, $P_{x'} - P_x = c + \sum_{i=1}^4 a_i\Sigma_i$ is a constant in $CH_2(F_1(Y))_{\mathbb{Q}}$, as desired. This concludes the proof of Proposition 4.3.29. □

Proof of Lemma 4.3.37. To check the irreducibility, we return to the proof of Lemma 4.3.36. Using the notation there, let

$$g(x_1,\dots,x_5) = f(0,x_1,\dots,x_5) = x_4Q_1(0,x_1,\dots,x_5) + x_5Q_2(0,x_1,\dots,x_5)$$

and let $q(x_1,\dots,x_5)$ be the degree 2 part of the polynomial $x_3^2 + x_4Q_1(1,x_1,\dots,x_5) + x_5Q_2(1,x_1,\dots,x_5)$. Then the surface Σ_1 of lines passing through the singular point $y = (1:0:0:0:0:0)$ is the subvariety in $\mathbb{P}^4 = \{(x_1:\dots:x_5)\}$ cut by the equations $g(x_1,\dots,x_5) = 0$ and $q(x_1,\dots,x_5) = 0$. From the expression, we see that $g(x_1,\dots,x_5)$ depends on the coefficients of the terms x_ix_j with $1 \le i,j \le 5$ in $Q(x_0,x_1,\dots,x_5)$ whereas $q(x_1,\dots,x_5)$ depends on the coefficients of the terms x_0x_i for $1 \le i \le 5$. So the choice of coefficients of $g(x_1,\dots,x_5)$ and $q(x_1,\dots,x_5)$ does not have influence on each other. Now we fix one choice of smooth $Q = \{q = 0\}$. Then varying $C = \{g = 0\}$, the base points of $C \cap Q$ is the line $(s:t:0:0:0)$ corresponding to the lines on the plane P_x passing through the point y. Hence, for a general choice of q and g, the surface $\Sigma_1 = C \cap Q$ is smooth outside the line $(s:t:0:0:0)$. Writing $Q_1(x_0,\dots,x_5) = \sum_{i,j=0}^5 a_{ij}x_ix_j$ and $Q_2(x_0,\dots,x_5) = \sum_{i,j=0}^5 b_{ij}x_ix_j$, the Jacobian matrix of the polynomials $q(x_1,\dots,x_5)$ and $g(x_1,\dots,x_5)$ at the point $(s:t:0:0:0)$ is

$$\begin{pmatrix} 0 & 0 & 0 & a_{01}s + a_{02}t & b_{01}s + b_{02}t \\ 0 & 0 & 0 & 2a_{12}st & 2b_{12}st \end{pmatrix}.$$

This matrix does not have full rank only if $s = 0$ or $t = 0$ or $(a_{01} - b_{01})s + (a_{02} - b_{02})t = 0$, corresponding to the three lines passing through the other three singular points. Therefore, the singular locus of Σ_1 is of codimension 2. But a reducible complete intersection of dimension ≥ 1 in the projective space always have codimension 1 singular locus by the Fulton–Hansen connectedness theorem [Ful79]. Hence, Σ_1 is irreducible, as desired. □

Proof of Lemma 4.3.38. Let H be the projective tangent space of Y at point y. It is a linear subspace of dimension 8 in $\mathbb{P}^9$. The linear section Y_4 being singular at y is equivalent to the relation $H_5 \subset H$. Let $F_{1,y}(Y)$ be the variety of lines in Y that passes through y. Let $\mathscr{P} = \{(l,y) \in F_1(Y) \times Y : y \in l\}$ be the incidence variety. Then $F_{1,y}(Y)$ can be identified with a subvariety of $\mathscr{P}$ given by the preimage of the point y under the projection map $q : \mathscr{P} \to Y$. Since every line in Y passing through y is contained in H, the subvariety S of $F_{1,y}(Y)$ parametrizing the lines that is furthermore included in H_5 is given by the zero locus of a section σ of the vector bundle $(\mathscr{E}^*)^{\oplus 3}$. However, as the point y already lies in H_5, if we restrict σ on y via the following morphism $(\mathscr{E}^*)^{\oplus 3} \to \mathscr{O}^{\oplus 3}$, we get zero. Thus, S can be viewed as the zero locus of a section of $(\det(\mathscr{E}^*))^3$, with expected dimension. Therefore, the Chow class of S in $F_1(Y)$ is given by $c_1(\mathscr{E}^*)^3 \cdot F_{1,y}(Y)$. Note that $F_{1,y}(Y) = \mathscr{P}^*(y) \in CH(F_1(Y))$ with $\mathscr{P}$ the canonical correspondence between $F_1(Y)$ and Y. Since $CH_0(Y)$ is trivial, the class $F_{1,y}(Y) \in CH(F_1(Y))$ is independent of y. In conclusion, the constant Chow class of S in $F_1(Y)$ is independent of the choice of H_5 and y. □

Chow Class of F in X

In this section, we determine the Chow class of F within X. Denote by $\mathscr{F}$ the tautological subbundle on $\mathrm{Gr}(4,10)$ (resp. $\mathrm{Gr}(r+2,n+1)$ in the general case), and $\mathscr{E}$ as the tautological subbundle of $X = F_2(Y)$ (resp. $X = F_r(Y)$ in the general case). Let c_i represent $c_i(\mathscr{E}^*)$ over X. In the case $r = 2$, we have

Proposition 4.3.39. *The Chow class of F within X is expressed as $-20c_1^3 + 110c_1c_2 + 49c_3$ in $CH^3(X)$.*

Proof. Let us start by giving the general method for any r and then specialize to $r = 2$ for explicit calculations. Consider the following stratification of $X \times \mathrm{Gr}(r+2,n+1)$:

$$\begin{array}{c} X \times \mathrm{Gr}(r+2,n+1) \\ \uparrow \gamma \\ \tilde{\tilde{X}} := \{(x,P_{r+1}) \in X \times \mathrm{Gr}(r+2,n+1) | P_{r+1} \cap Y \supset P_x\} \\ \uparrow \beta \\ \tilde{X} := \{(x,P_{r+1}) \in X \times \mathrm{Gr}(r+2,n+1) | P_{r+1} \cap Y \supset 2P_x\} \\ \uparrow \alpha \\ \tilde{F} := \{(x,P_{r+1}) \in X \times \mathrm{Gr}(r+2,n+1) | P_{r+1} \cap Y \supset 3P_x\} \end{array}$$

Let $pr_1 : X \times \mathrm{Gr}(r+2,n+1) \to X$ and $pr_2 : X \times \mathrm{Gr}(r+2,n+1) \to \mathrm{Gr}(r+2,n+1)$ be the projection maps. It is not hard to see that $F = (pr_1 \circ \gamma \circ \beta \circ \alpha)(\tilde{F})$. We will see shortly that the Chow class of $\tilde{F}$ in $X \times \mathrm{Gr}(r+2,n+1)$ lies in a subring generated by the Chern classes of $pr_1^*\mathscr{E}^*$ and $pr_2^*\mathscr{F}^*$, say

$$[\tilde{F}] = \sum_i pr_1^* P_i(c_k(\mathscr{E}^*)) \cdot pr_2^* Q_i(c_l(\mathscr{F}^*)) \tag{4.21}$$

in $CH^{r+1+(n-r-1)(r+2)}(X \times \mathrm{Gr}(r+2,n+1))$, where P_i and Q_i are polynomials with adequate degrees. Then the class of F in X is $pr_{1*}([\tilde{F}]) = \sum_i P_i(c_k(\mathscr{E}^*)) \cdot pr_{1*}pr_2^* Q_i(c_l(\mathscr{F}^*))$. In this expression, $pr_{1*}pr_2^* Q_i(c_l(\mathscr{F}^*)) \neq 0$ only if the weighted degree of $Q_i(c_l(\mathscr{F}^*))$ is $(n-r-1)(r+2)$, namely the dimension of $\mathrm{Gr}(r+2,n+1)$ (or equivalently, the weignted degree of $P_i(c_k(\mathscr{E}^*))$ is $r+1$). In this case, in $CH_0(\mathrm{Gr}(r+2,n+1))$, the class $Q_i(c_l(\mathscr{F}^*))$ is a multiple of a point $o \in \mathrm{Gr}(r+2,n+1)$. The coefficient, denoted by q_i, is calculable using Schubert calculus, once we know the expression of $Q_i(c_l(\mathscr{F}^*))$. Taken together, the class of F in $CH^{r+1}(X)$ is given by

$$[F] = \sum_{i:\deg P_i = r+1} q_i P_i(c_k(\mathscr{E}^*)).$$

Now let us prove (4.21), i.e., $[\tilde{F}]$ is generated by $pr_1^*\mathscr{E}^*$ and $pr_2^*\mathscr{F}^*$. The subvariety $\tilde{\tilde{X}}$ of $X \times \mathrm{Gr}(r+2,n+1)$ is defined by the condition $P_x \subset P_{r+1}$. Hence, we can view $\tilde{\tilde{X}}$ as the locus where the composite map of vector bundles $pr_1^*\mathscr{E} \to V_{n+1} \to V_{n+1}/pr_2^*\mathscr{F}$ is zero, where V_{n+1} is the trivial bundle of rank $n+1$ over $X \times \mathrm{Gr}(r+2,n+1)$. Hence, $\tilde{\tilde{X}}$ is the zero locus of a section of $pr_1^*\mathscr{E}^* \otimes V_{n+1}/pr_2^*\mathscr{F}$. Now over $\tilde{\tilde{X}}$, we have a natural injection $pr_1^*\mathscr{E}|_{\tilde{\tilde{X}}} \to pr_2^*\mathscr{F}|_{\tilde{\tilde{X}}}$. The defining polynomial f of Y induces a section of $pr_2^*\mathrm{Sym}^3\mathscr{F}^*|_{\tilde{\tilde{X}}}$.

Since $P_x \subset Y$ for any $x \in X$, this section vanishes when passing to $pr_1^*\mathrm{Sym}^3\mathscr{E}^*|_{\tilde{\tilde{X}}}$. Hence, the section is actually a section $\tilde{\tilde{f}}$ of $pr_2^*\mathrm{Sym}^2\mathscr{F}^*|_{\tilde{\tilde{X}}} \otimes (pr_2^*\mathscr{F}|_{\tilde{\tilde{X}}}/pr_1^*\mathscr{E}|_{\tilde{\tilde{X}}})^*$. The subvariety $\tilde{X}$ is the locus of $\tilde{\tilde{X}}$ where, when passing to $pr_1^*\mathrm{Sym}^2\mathscr{E}^*|_{\tilde{\tilde{X}}} \otimes (pr_2^*\mathscr{F}|_{\tilde{\tilde{X}}}/pr_1^*\mathscr{E}|_{\tilde{\tilde{X}}})^*$, the section $\tilde{\tilde{f}}$ is zero. Hence, $\tilde{X}$ is the zero locus of a section of the vector bundle $pr_1^*\mathrm{Sym}^2\mathscr{E}^*|_{\tilde{\tilde{X}}} \otimes (pr_2^*\mathscr{F}|_{\tilde{\tilde{X}}}/pr_1^*\mathscr{E}|_{\tilde{\tilde{X}}})^*$. Similar argument shows that $\tilde{F}$ is the zero locus of a section of the vector bundle $pr_1^*\mathscr{E}^*|_{\tilde{X}} \otimes (pr_2^*\mathscr{F}|_{\tilde{X}}/pr_1^*\mathscr{E}|_{\tilde{X}})^{*\otimes 2}$. Taken together, the class of $\tilde{F}$ in $CH(X \times \mathrm{Gr}(r+2,n+1))$ is given by the following class

$$e(pr_1^*\mathscr{E}^* \otimes (V_{n+1} - pr_2^*\mathscr{F})) \cdot e((pr_2^*\mathscr{F}^* - pr_1^*\mathscr{E}^*) \otimes pr_1^*\mathrm{Sym}^2\mathscr{E}^*) \cdot e((pr_2^*\mathscr{F}^* - pr_1^*\mathscr{E}^*)^{\otimes 2} \otimes pr_1^*\mathscr{E}^*)$$

where $e(...)$ is the Euler class and we have written the vector bundles as their class in K groups to avoid any confusions. This gives an expression of the class $\tilde{F}$ as generated by the Chern classes of $pr_1^*\mathscr{E}^*$ and $pr_2^*\mathscr{F}^*$.

In what follows, we do the explicit calculations in the case $r = 2$, with the help of Mathematica and SageMath.

Calculations of $e(pr_1^*\mathscr{E}^* \otimes (V_{10} - pr_2^*\mathscr{F}))$

We may assume that $pr_1^*\mathscr{E}^*$ is formally a direct sum of three line bundles whose classes are l_1, l_2, l_3 and that $V_{10}/pr_2^*\mathscr{F}$ is formally a direct sum of $m_1, \ldots, m_6$. Then the Euler class of $pr_1^*\mathscr{E}^* \otimes V_{10}/pr_2^*\mathscr{F}$ is $\prod_{i,j}(l_i + m_j)$. The Euler class of $pr_1^*\mathscr{E}^* \otimes V_{10}/pr_2^*\mathscr{F}$ expressed by c_i, the Chern classes of $pr_1^*\mathscr{E}^*$, and by d_j, the Chern classes of $V_{10}/pr_2^*\mathscr{F}$ is calculated, using Mathematica, in Appendix A.1.1. The result is tremendously long. Fortunately, we do not really need the full form of the result. As our general method indicates, we only need to expand the Chern classes of $pr_1^*\mathscr{E}^*$ up to degree 3. With this in mind, the Euler class of $pr_1^*\mathscr{E}^* \otimes (V_{10} - pr_2^*\mathscr{F})$ is

```
d6^3 + c1 d5 d6^2 + c1^2 d4 d6^2 + c2 (d5^2 d6 - 2 d4 d6^2) +
c1 c2 (d4 d5 d6 - 3 d3 d6^2) + c1^3 d3 d6^2 +
c3 (d5^3 - 6 d3 d6^2 + 3 (-d4 d5 d6 + 3 d3 d6^2))+ higher terms in ci
```

Calculations of $e((pr_2^*\mathscr{F}^* - pr_1^*\mathscr{E}^*) \otimes pr_1^*\mathrm{Sym}^2\mathscr{E}^*)$ and $e(((pr_2^*\mathscr{F}^* - pr_1^*\mathscr{E}^*)^{\otimes 2} \otimes pr_1^*\mathscr{E}^*)$

Let $C_i = c_i(pr_2^*\mathscr{F}^*)$ be the Chern classes of $pr_2^*\mathscr{F}^*$ and let $S_i = c_i(pr_1^*\mathrm{Sym}^2\mathscr{E}^*)$. Then S_i can be represented by the c_i, the Chern classes of $pr_1^*\mathscr{E}^*$ as follows.

```
S1[c1_, c2_, c3_] := 4 c1;
S2[c1_, c2_, c3_] := 5 c1^2 + 5 c2;
S3[c1_, c2_, c3_] := 2 c1^3 + 11 c1 c2 + 7 c3;
```

It is a general fact [Ful84, Example 3.2.2] that if L is a line bundle and V is a vector bundle of rank s, then

$$e(L \otimes V) = \sum_{i=0}^{s} c_1(L)^i c_{s-i}(V).$$

Since $c_1(pr_2^*\mathscr{F}^* - pr_1^*\mathscr{E}^*) = c_1(pr_2^*\mathscr{F}^*) - c_1(pr_1^*\mathscr{E}^*)$, the class $e((pr_2^*\mathscr{F}^* - pr_1^*\mathscr{E}^*) \otimes pr_1^*\mathrm{Sym}^2\mathscr{E}^*)$ is given by the expression

```
(C1 - c1)^6 + (C1 - c1)^5 S1[c1, c2, c3] + (C1 - c1)^4 S2[c1, c2, c3]
+ (C1 - c1)^3 S3[c1, c2, c3]+ higher terms in ci
```

and the class $e(((pr_2^*\mathscr{F}^* - pr_1^*\mathscr{E}^*)^{\otimes 2} \otimes pr_1^*\mathscr{E}^*)$ is given by

```
8 (C1 - c1)^3 + 4 (C1 - c1)^2 c1 + 2 (C1 - c1) c2 + c3.
```

Calculation of $[F]$

Now we can calculate $e(pr_1^*\mathscr{E}^* \otimes (V_{n+1} - pr_2^*\mathscr{F})) \cdot e((pr_2^*\mathscr{F}^* - pr_1^*\mathscr{E}^*) \otimes pr_1^*\mathrm{Sym}^2\mathscr{E}^*) \cdot e(((pr_2^*\mathscr{F}^* - pr_1^*\mathscr{E}^*)^{\otimes 2} \otimes pr_1^*\mathscr{E}^*)$ (up to degree 3 in c_i). The detailed Mathematica calculations and results are presented in Appendix A.1.2. We only need the expression up to degree 3 in c_i. The expression of $[\tilde{F}]$ is

```
c1^3 (8 C1^9 d3 d6^2 - 36 C1^8 d4 d6^2 + 56 C1^7 d5 d6^2
        - 20 C1^6 d6^3)+
c3 (8 C1^9 d5^3 - 24 C1^9 d4 d5 d6 + 24 C1^9 d3 d6^2 + 57 C1^6 d6^3)+
c1 c2 (8 C1^9 d4 d5 d6 - 36 C1^8 d5^2 d6 - 24 C1^9 d3 d6^2 +
        72 C1^8 d4 d6^2 + 42 C1^7 d5 d6^2 - 178 C1^6 d6^3)+
+ higher terms in ci.
```

Now we need to calculate the coefficients in the above expression. To this ends, we do the Schubert calculus with the help of SageMath. We replace C_i by $s[1,...,1]$ with i copies of 1 and replace d_i by $(-1)^i s[i]$, where $s[1,...,1]$ and $s[i]$ are the Schur's polynomials with the corresponding weights. We do the Schubert calculus using Schur's polynomials and find the coefficients of $s[6,6,6,6]$. The detailed SageMath calculations and results are presented in Appendix A.1.3. The output result is exactly $-20c_1^3 + 110c_1c_2 + 49c_3$, as desired. □

Remark 4.3.40. Using the same argument (but with much simpler calculations), we find that for the Fano variety of lines of cubic fourfold, the fixed locus of the Voisin map is $21c_2$—a result that coincides with the result of [GK20, Theorem A].

4.4 Indeterminacy Locus of the Voisin Map

In this section, we explore the indeterminacy locus of the Voisin map $\Psi : X \dashrightarrow X$. As previously discussed in Section 4.1.3, the indeterminacy locus comprises two components, described as follows. The first component, Ind_0, parametrizes points $x \in X$ that represent $P_x \subset Y$, where there are more than one linear subspaces of dimension $r+1$ tangent to Y at P_x. The second component, Ind_1, includes points $x \in X$ representing $P_x \subset Y$, where an $(r+1)$-dimensional linear space both contains P_x and is contained within Y. The main purpose of this section is to prove Theorem C.

4.4.1 Proof of Theorem C

Let us analyse the action of the Voisin map Ψ on the Chern classes of X and on the divisor classes of X respectively. Note that for $r \geq 2$, $CH^1(X)$ is generated by only one divisor h, which is the restriction of Plücker line bundle on $\mathrm{Gr}(r+1, n+1)$.

Lemma 4.4.1. *For any $r \geq 1$, we have $\Psi^* h = (3r+4)h$.*

Proof. Let $\mathscr{E}$ be the tautological subbundle of $X \subset \mathrm{Gr}(r+1, n+1)$. This is a rank $r+1$ subbundle of the trivial vector bundle $V_{n+1} \otimes \mathscr{O}_X$. Let $\mathscr{F}$ be the kernel of the following

morphism of $\mathscr{O}_X$-modules

$$\begin{array}{rccc} \mathscr{G}: & V_{n+1}\otimes\mathscr{O}_X & \to & \mathscr{S}ym^2\mathscr{E}^* \\ & (a_0,\dots,a_n) & \mapsto & (\sum_i a_i\frac{\partial f}{\partial x_i}|_{P_x})_{x\in X}. \end{array}$$

The codimension of $\mathrm{Ind}_0\subset X$ is 2. Hence, the result will not change if we replace X by $X':=X-\mathrm{Ind}_0$. We use the same notations for the $\mathscr{O}_X$-modules restricted to X'. Then $\mathscr{F}$ becomes a vector bundle of rank $r+2$ on X' which fits in the following short exact sequence

$$0\to\mathscr{F}\to V_{n+1}\otimes\mathscr{O}_X\to\mathrm{Sym}^2\mathscr{E}^*\to 0. \tag{4.22}$$

From this short exact sequence, we deduce that

$$c_1(\mathscr{F})=-(r+2)h. \tag{4.23}$$

The homogeneous polynomial f induces a section σ_f of the vector bundle $\mathrm{Sym}^3\mathscr{F}^*$, and the fact that $\mathbb{P}(\mathscr{F}|_x)$ is the $(r+1)$-space that is tangent to Y along P_x shows that σ_f can be viewed as an element in $H^0(X',(\mathscr{F}/\mathscr{E})^*\otimes(\mathscr{F}/\mathscr{E})^*\otimes\mathscr{F}^*)$ via the following short exact sequences

$$0\to(\mathscr{F}/\mathscr{E})^*\otimes\mathrm{Sym}^2\mathscr{F}^*\to\mathrm{Sym}^3\mathscr{F}^*\to\mathrm{Sym}^3\mathscr{E}^*\to 0,$$

$$0\to(\mathscr{F}/\mathscr{E})^*\otimes(\mathscr{F}/\mathscr{E})^*\otimes\mathscr{F}^*\to(\mathscr{F}/\mathscr{E})^*\otimes\mathrm{Sym}^2\mathscr{F}^*\to(\mathscr{F}/\mathscr{E})^*\otimes\mathrm{Sym}^2\mathscr{E}^*\to 0.$$

Therefore, σ_f induces a morphism $\phi:\mathscr{F}\to(\mathscr{F}/\mathscr{E})^*\otimes(\mathscr{F}/\mathscr{E})^*$. Then the locus of X where ϕ is not surjective is Ind_1. By Proposition 4.4.5 proved in Section 4.4.2, the codimension of Ind_1 in X is $r+2$. Therefore, the result will not change if we replace X' by $X''=X-\mathrm{Ind}_0-\mathrm{Ind}_1$ and we use the same notations for the restrictions of $\mathscr{O}_X$-modules to X''. On X'', the vector bundle $\Psi^*\mathscr{E}$ is exactly the kernel of ϕ and $\Psi^*\mathscr{E}$ fits into the following short exact sequence

$$0\to\Psi^*\mathscr{E}\to\mathscr{F}\to(\mathscr{F}/\mathscr{E})^*\otimes(\mathscr{F}/\mathscr{E})^*\to 0.$$

Therefore, $\Psi^*c_1(\mathscr{E})=c_1(\Psi^*\mathscr{E})=c_1(\mathscr{F})-2c_1((\mathscr{F}/\mathscr{E})^*)=-(r+2)h-2(-(-(r+2)h+h))=-(3r+4)h$. Since $c_1(\mathscr{E})=-h$, we find that $\Psi^*h=(3r+4)h$, as desired. □

Remark 4.4.2. When $r=1$ and $n=5$, Y is a cubic fourfold and X is a hyper-Kähler fourfold. Our result recovers the result of [Ame09] and [SV15, Proposition 21.4] which states $\Psi^*h=7h$.

Lemma 4.4.3. *Let $X^0=X-\mathrm{Ind}$ be the locus where X is defined. We have $(\Psi^*T_X)|_{X^0}=T_{X^0}$.*

Proof. This result has been implicitly proven in [Voi04]. Let $\tau:\tilde{X}\to X$, $\tilde{\Psi}:\tilde{X}\to X$ be the desingularisation of the indeterminacies of the Voisin map $\Psi:X\dashrightarrow X$. Since X is a K-trivial variety, the exceptional divisor of τ coincides with the ramification locus of $\tilde{\Psi}$ (see [Voi04, Lemma 4]). Therefore, the Voisin map restricted to the defined domain $\Psi|_{X^0}:X^0\to X$ is étale, which implies that the relative tangent bundle $T_{X^0/X}$ by the map $\Psi|_{X^0}$ is zero. Hence the result. □

Proof of Theorem C. We only consider the case $r=2$. Let $M\in CH_0(X)_{hom}$ be a class belonging to the subring generated by h and by $c_i(X)$. Let us assume M is a monomial of the form $h^k\prod c_i(X)$. By Lemma 4.4.1 and Lemma 4.4.3, we have

$$(\Psi^*M)|_{X^0}=10^k\cdot M|_{X^0}, \tag{4.24}$$

where $X^0 = X - \mathrm{Ind}$ is the locus where Ψ is defined. On the other hand, by Theorem B, we have

$$\Psi^* M = -8M. \tag{4.25}$$

By (4.24) and (4.25), we find that $M|_{X^0} = 0$, which means that M is supported on Ind, as desired. □

4.4.2 Some Numerical Data

Proposition 4.4.4. *The first component,* Ind_0*, has a codimension of* 2 *in X, and its Chow class in X is expressed as:*

$$\left(\frac{1}{2}(r+2)(r+1)+2\right)c_1^2 - (r+4)c_2.$$

Proof. With the same notations as in the proof of Lemma 4.4.1 in Section 4.4.1, Then at point $x \in X$, the projectivation of the vector space $\mathscr{F}|_x$ is the intersection of the projective tangent spaces $T_{Y,y}$ for all $y \in P_x$. We conclude that Ind_0 is the locus in X where $\mathscr{G}$ is not surjective. The map $\mathscr{G}$ sends the subbundle $\mathscr{E}$ to 0. Hence, $\mathscr{G}$ factorizes through the following morphism

$$\bar{\mathscr{G}} : (V_{n+1} \otimes \mathscr{O}_X)/\mathscr{E} \to \mathrm{Sym}^2 \mathscr{E}^*.$$

For Y general, the locus where $\bar{\mathscr{G}}$ is not surjective is of codimension $\mathrm{rank}((V_{n+1} \otimes \mathscr{O}_X)/\mathscr{E}) - \mathrm{rank}(\mathrm{Sym}^2 \mathscr{E}^*) + 1$, which equals 2 with the relation $n+1 = \binom{r+3}{2}$ taken into account.

Now let us calculate the Chow class of Ind_0 in X. We have viewed Ind_0 as the locus where the map $\bar{\mathscr{G}} : (V_{n+1} \otimes \mathscr{O}_X)/\mathscr{E} \to \mathrm{Sym}^2 \mathscr{E}^*$ is not surjective. By the Porteous formula (see the Introduction part of [HT84]), the class of Ind_0 is represented by the degree 2 part of

$$\frac{c((V_{n+1} \otimes \mathscr{O}_X)/\mathscr{E})}{c(\mathrm{Sym}^2 \mathscr{E}^*)} = \frac{1}{c(\mathscr{E})c(\mathrm{Sym}^2 \mathscr{E}^*)}.$$

Suppose $\mathscr{E}^*$ formally splits into $l_1, \ldots, l_{r+1}$. Then

$$c(\mathrm{Sym}^2 \mathscr{E}^*) = \prod_{1 \le i \le j \le r+1} (1 + l_i + l_j).$$

The degree 1 part of this expression is given by $\sum_{1 \le i \le j \le r+1}(l_i + l_j) = (r+2)c_1$. The degree 2 part of this expression is given by $\frac{1}{2}\sum_{(i,j) \ne (k,l) \in \Delta_{r+1}}(l_i + l_j)(l_k + l_l)$, where $\Delta_{r+1} = \{(i,j) : 1 \le i \le j \le r+1\}$. We calculate

$$\begin{aligned}
&\sum_{(i,j) \ne (k,l) \in \Delta_{r+1}} (l_i + l_j)(l_k + l_l) \\
=&\sum_{(i,j),(k,l) \in \Delta_{r+1}} (l_i + l_j)(l_k + l_l) - \sum_{(i,j) \in \Delta_{r+1}} (l_i + l_j)^2 \\
=&(\sum_{(i,j) \in \Delta_{r+1}} (l_i + l_j))^2 - \sum_{(i,j) \in \Delta_{r+1}} (l_i^2 + l_j^2) - 2\sum_{(i,j) \in \Delta_{r+1}} l_i l_j \\
=&(r+2)^2 c_1^2 - (r+4)\sum_i l_i^2 - 2\sum_{i<j} l_i l_j \\
=&(r+2)^2 c_1^2 - (r+4)(c_1^2 - 2c_2) - 2c_2 \\
=&(r^2+3r)c_1^2 + 2(r+3)c_2.
\end{aligned}$$

Hence,

$$c(\mathrm{Sym}^2\mathscr{E}^*) = 1+(r+2)c_1+\frac{1}{2}(r^2+3r)c_1^2+(r+3)c_2+\dots$$

On the other hand, we have $c(\mathscr{E}) = 1-c_1+c_2+\dots$. Finally, by a formal calculation, we find that the degree 2 part of the expression $\frac{1}{c(\mathscr{E})c(\mathrm{Sym}^2\mathscr{E}^*)}$ is given by $(\frac{1}{2}(r+2)(r+1)+2)c_1^2-(r+4)c_2$, as desired. □

Proposition 4.4.5. *The second component,* Ind_1*, has a codimension of* $r+2$ *in* X *for* $r\geq 2$*. For* $r\leq 1$*, the set* Ind_1 *is empty.*

Proof. Recall that Ind_1 is described as the locus of points $x\in X$ such that there is an $r+1$ space in Y containing P_x. For $r\geq 2$, the variety $F_{r+1}(Y)$ is nonempty. There is a natural incidence variety

$$\mathscr{P}_{r+1,r} := \{(\lambda,x)\in F_{r+1}(Y)\times F_r(Y) : P_\lambda \supset P_x\}.$$

The variety $\mathscr{P}_{r+1,r}$ is a $\mathbb{P}^{r+1}$-bundle over $F_{r+1}(Y)$, so $\dim\mathscr{P}_{r+1,r} = \dim F_{r+1}(Y)+(r+1)$. It is clear that Ind_1 is the image of the second projection map $\mathscr{P}_{r+1,r}\to F_r(Y)$. Hence, $\mathrm{codim}\,\mathrm{Ind}_1 = \dim F_r(Y)-\dim F_{r+1}(Y)-(r+1) = r+2$. For $r\leq 1$, the variety $F_{r+1}(Y)$ is empty. □

4.4.3 Discussions on Question 4.1.17

In this part, we would like to give some evidence on Question 4.1.17. Let $X = F_r(Y)$ be the strict Calabi-Yau manifold constructed as in Section 4.1.1 for any $r\geq 2$. Let $\mathscr{P} = \{(x,y)\in X\times Y : y\in P_x\}$ be the incidence correspondence of X and Y. Let $\tau:\tilde{X}\to X$, $\tilde{\Psi}:\tilde{X}\to X$ be the desingularization of the indeterminacy locus of the Voisin map $\Psi: X\dashrightarrow X$.

Proposition 4.4.6. *If the induced map*

$$\mathscr{P}_* : CH_1(X)_{hom}\to CH_{r+1}(Y)_{hom}$$

is injective, then the divisor $\tilde{\Psi}(\tau^{-1}(\mathrm{Ind}_0))$ *is a constant-cycle subvariety in* X*.*

Proof. Let $x\in Ind_0$ be a general point representing an r-linear subspace P_x in Y. Then there is a unique $(r+2)$-linear subspace Θ_x that is tangent to Y along P_x. Then the preimage $\tau^{-1}(x)$ is a rational curve $\mathbb{P}^1_x$ parametrizing the $(r+1)$-linear subspaces H containing P_x and contained in Θ_x. Each H intersects Y with a double P_x and a residual r-linear subspace $P_{x'}$ represented by a point $x'\in X$. The image $\Gamma_x := \tilde{\Psi}(\mathbb{P}^1_x)$ is the rational curve in X parametrizing all these residual planes $P_{x'}$. Applying the correspondence $\mathscr{P}_*$ on Γ_x, it is not hard to see that $\mathscr{P}_*(\Gamma_x)$ is the algebraic cycle represented by $\Theta_x\cap Y$, and that this class is independent of the choice of $x\in\mathrm{Ind}_0$ since it is a linear section of Y. By our assumption that $\mathscr{P}_* : CH_1(X)_{hom}\to CH_{r+1}(Y)_{hom}$, the Chow classes of the rational curves $\Gamma_x\in X$ are independent of the choice of $x\in\mathrm{Ind}_0$. Notice that the divisor $\tilde{\Psi}(\tau^{-1}(\mathrm{Ind}_0))$ is the closure of the union of these rational curves Γ_x. Let $y_x\in\Gamma_x$ and $y_{x'}\in\Gamma_{x'}$ and let $D\subset X$ be an ample divisor such that $d = \deg(D.\Gamma_x) = \deg(D.\Gamma_{x'})$. Then $d.y_x = D.\Gamma_x = D.\Gamma_{x'} = d.y_{x'}$. Since $CH_0(X)_{hom}$ is torsion free, we find that $y_x = y_{x'}$ in $CH_0(X)$, as desired. □

Now let us focus on the case $r=2$.

Proposition 4.4.7. *In the case $r = 2$, if the induced map*

$$\mathscr{P}_* : CH_1(X)_{hom} \to CH_3(Y)_{hom}$$

is injective, then the first component Ind_0 *of the indeterminacy locus is a constant-cycle subvariety.*

Proof. In the case $r = 2$, Theorem B implies that $\Psi_*(x_1 - x_2) = -8(x_1 - x_2)$. But Proposition 4.4.6 implies that $\Psi_*(x_1 - x_2) = 0$ since the divisor $\tilde{\Psi}(\tau^{-1}(\mathrm{Ind}_0))$ is a constant-cycle subvariety in X. Therefore, $x_1 = x_2 \in CH_0(X)$ since $CH_0(X)$ is torsion-free. □

Remark 4.4.8. If we assume Conjecture 4.1.12, and if we suppose that the induced map

$$\mathscr{P}_* : CH_1(X)_{hom} \to CH_{r+1}(Y)_{hom}$$

is injective, then using the same argument as in Proposition 4.4.7, we can actually show that Ind_0 is a constant-cycle subvariety for any $r \geq 2$. Notice that $\mathscr{P}_* : CH_1(X)_{hom} \to CH_2(Y)_{hom}$ is *not* injective in the hyper-Kähler case, since in that case, we know that $CH_2(Y)_{hom} = 0$ (see [BS83, Theorem 1 (ii)]) and [Huy24, Section 5.0.1] while $CH_1(F_1(Y))_{hom}$ is big (see [SV15, Theorem 21.9]). This phenomenon is also reflected by the fact that there is not any constant-cycle divisor in a hyper-Kähler manifold of dimension ≥ 4 (see [Voi16]) and that Ind_0 is not a constant-cycle subvariety in this case (see [Ame09, Lemma 1], [Voi04]).

In the spirit of Proposition 4.4.6 and Proposition 4.4.7, it is reasonable to propose the following

Question 4.4.9. If $r \geq 2$, is the induced map

$$\mathscr{P}_* : CH_1(X)_{hom} \to CH_{r+1}(Y)_{hom}$$

injective?

In alignment with the principles of the generalized Bloch conjecture and the generalized Hodge conjecture, the Chow group of 1-cycles on a smooth projective variety X is expected to be "governed" by the quotient $N^1/N^2H^*(X,\mathbb{Q})$, where N^i denotes the Hodge coniveau filtration on the cohomology $H^*(X,\mathbb{Q})$. Given that X is a strict Calabi-Yau manifold, the derivative of the period map

$$\mathscr{D} : \mathrm{Def}(X) \to \mathbb{P}H^N(X,\mathbb{C}),$$

that associates any point $b \in \mathrm{Def}(X)$ with the 1-dimensional subspace $H^{N,0}(X_b) \subset H^N(X_b,\mathbb{C}) \cong H^N(X,\mathbb{C})$, has as its image

$$\mathrm{Hom}(H^{N,0}(X), H^{N-1,1}(X)) \subset \mathrm{Hom}(H^{N,0}(X), H^N(X,\mathbb{C})/H^{N-1,1}(X)),$$

according to Griffiths' theory on period maps. Question 4.4.9 would likely have an affirmative answer if the following conditions are met:

- The map $F_r : \mathbb{P}H^0(\mathbb{P}^n, \mathscr{O}_{\mathbb{P}^n}(3)) \dashrightarrow \mathrm{Def}(X)$ is dominant, implying that for a general Y, the Fano variety of r-linear spaces within Y, denoted by X, is also general in the moduli space of X. Consequently, $N^1/N^2H^N(X,\mathbb{Q}) = 0$.

- The induced map $[P]^* : H^{p+r,1+r}(Y) \to H^{p,1}(X)$ is surjective for $1 < p < N$. Considering that $[P]^*$ is injective (refer to, for example, [Voi14, Lemma 4.6]), it is sufficient to compute the dimension of $H^{p,1}(X)$ and compare it to that of $H^{p+1,1+r}(Y)$. Notice that the dimension of $H^{p+1,1+r}(Y)$ is well-established (see, for example, [Huy24, Section 1.1] or [Voi03, Chapitre 18]).

The following provides some (partial) affirmative responses regarding the two aforementioned desired conditions.

Theorem 4.4.10. *Let $X = F_r(Y)$ be the strict Calabi-Yau manifold as described in Section 4.1.1, with $r \geq 2$, then the map $F_r : \mathbb{P}H^0(\mathbb{P}^n, \mathscr{O}_{\mathbb{P}^n}(3)) \dashrightarrow \mathrm{Def}(X)$, which assigns to $Y \in \mathbb{P}H^0(\mathbb{P}^n, \mathscr{O}_{\mathbb{P}^n}(3))$ its Fano variety of r-spaces $F_r(Y)$, is dominant and has a relative dimension of $n^2 + 2n = \dim \mathrm{PSL}_{n+1}(\mathbb{C})$.*

Remark 4.4.11. Theorem 4.4.10 does not apply if $r = 1$, due to the existence of non-projective deformations of hyper-Kähler manifolds. This limitation is further highlighted by our proof of Theorem 4.4.10, which crucially relies on the condition that $r \geq 2$.

For the explicit calculations of Hodge numbers $h^{p,1}(X)$, we only manage to get the result in the case $r = 2$ due to the complexity of the computation.

Proposition 4.4.12. *In the case of $r = 2$, we have the following*

$$h^{p,1}(X) = \begin{cases} 1, & p = 1 \\ 45, & p = 3 \\ 120, & p = 10 \\ 0, & \textit{others}. \end{cases}$$

Notice that for $p > 1$, the only non-zero $H^{p+2,3}(Y)$ for a cubic eightfold Y is $H^{5,3}(Y)$ whose dimension is 45 (see [Huy24, Section 1.1]). Thus, Proposition 4.4.12 indeed confirms the second condition mentioned above in the case of $r = 2$.

The proofs of Theorem 4.4.10 and Proposition 4.4.12 are detailed in the following sections.

Proof of Theorem 4.4.10

Proof of Theorem 4.4.10. For the brevity of the notation, in what follows, we will denote G as the Grassmannian $\mathrm{Gr}(r+1, n+1)$ and $S^3\mathscr{E}^*$ as the symmetric product $\mathrm{Sym}^3\mathscr{E}^*$. Let

$$\alpha : H^0(\mathbb{P}^n, \mathscr{O}_{\mathbb{P}^n}(3)) \cong H^0(G, S^3\mathscr{E}^*) \to H^0(X, S^3\mathscr{E}^*|_X) \cong H^0(X, N_{X/G})$$

be the restriction map composed with some canonical isomorphisms. Let

$$\beta : H^0(X, N_{X/G}) \to H^1(X, T_X)$$

be the connection map of the normal exact sequence. With the above notations, we have the following two lemmas that we will prove later.

Lemma 4.4.13. *$H^0(X, N_{X/G})$ is identified with the tangent space of $\mathbb{P}H^0(\mathbb{P}^n, \mathscr{O}_{\mathbb{P}^n}(3))$ at the point f representing the cubic hypersurface Y_0.*

Lemma 4.4.14. *The map β is surjective and $\dim \ker \beta = n^2 + 2n$.*

Therefore, the map $\beta : H^0(X, N_{X/G}) \to H^1(X, T_X)$ identifies with the tangent map of $F_r : \mathbb{P}H^0(\mathbb{P}^n, \mathscr{O}_{\mathbb{P}^n}(3)) \dashrightarrow \mathrm{Def}(X))$, and Theorem 4.4.10 follows, knowing that $\mathrm{Def}(X)$ is smooth since X is a strict Calabi-Yau manifold (see [**?**, **?**, **?**] and [Ran92]). □

The proof of Lemma 4.4.13 and of Lemma 4.4.14 is a direct calculation using the following theorem of Bott [Bot75, Dem76]. We use the following formulation presented in [Man92, Proposition 2]. In the statement, L_λ represents the Schur's functor of weight λ.

Theorem 4.4.15 (Bott [Bot75, Dem76, Man92])**.** *Given decreasing weights $\lambda_1 \in \mathbb{Z}^{n-r}$ and $\lambda_2 \in \mathbb{Z}^{r+1}$. Let $\lambda = (\lambda_1, \lambda_2) \in \mathbb{Z}^{n+1}$. Let $c(n+1) = (1, 2, \ldots, n+1)$. If $\lambda - c(n+1)$ has common components, then $H^q(\mathrm{Gr}(r+1, n+1), L_{\lambda_1}\mathscr{Q} \otimes L_{\lambda_2}\mathscr{E}) = 0$ for any q (and in this case, we say that $\lambda - c(n+1)$ is irregular). Otherwise, we have*

$$H^q(\mathrm{Gr}(r+1, n+1), L_{\lambda_1}\mathscr{Q} \otimes L_{\lambda_2}\mathscr{E}) = \delta_{q,i(\lambda)} L_{\xi(\lambda)} V_{n+1}.$$

Here $i(\lambda)$, $\xi(\lambda)$ are defined as follows. The weight $\xi(\lambda)$ is defined as $(\lambda - c(n+1))^{\geq} + c(n+1)$, where $(\lambda - c(n+1))^{\geq}$ is the decreasing integer sequence obtained by permutation of the sequence $\lambda - c(n+1)$. The number $i(\lambda)$ is the inversion number of the sequence $\lambda - c(n+1)$.

The following technical lemma, due to Debarre and Manivel [DM98, Lemma 3.9], is also used frequently.

Lemma 4.4.16 (Debarre-Manivel [DM98])**.** *Let V be a complex vector space, m and d be integers. For any irreducible component $L_\lambda V$ of $\bigwedge^j \mathrm{Sym}^d V$, we have*

$$|\lambda|_{>m} \geq j - \binom{m+d-1}{d}.$$

Here, we write λ as a decreasing sequence of integers $(\lambda_1, \ldots, \lambda_{\dim V})$ and $|\lambda|_{>m} = \sum_{i>m} \lambda_i$.

Proof of Lemma 4.4.13. Let $\mathscr{I}_X$ be the ideal sheaf of X in G that fits into the following short exact sequence

$$0 \to \mathscr{I}_X \otimes S^3\mathscr{E}^* \to S^3\mathscr{E}^* \to N_{X/G} \to 0.$$

To prove Lemma 4.4.13, it suffices to prove that $H^0(G, \mathscr{I}_X \otimes S^3\mathscr{E}^*)$ is of dimension 1 while $H^1(G, \mathscr{I}_X \otimes S^3\mathscr{E}^*) = 0$. By the Koszul resolution

$$\ldots \to \bigwedge^{i+1} S^3\mathscr{E} \otimes S^3\mathscr{E}^* \to \bigwedge^{i} S^3\mathscr{E} \otimes S^3\mathscr{E}^* \to \ldots \to S^3\mathscr{E} \otimes S^3\mathscr{E}^* \to \mathscr{I}_X \otimes S^3\mathscr{E}^* \to 0,$$

it suffices to prove

(i) $\dim H^0(G, S^3\mathscr{E} \otimes S^3\mathscr{E}^*) = 1$ and $H^1(G, S^3\mathscr{E} \otimes S^3\mathscr{E}^*) = 0$.

(ii) $H^{i-d}(G, \bigwedge^i S^3\mathscr{E} \otimes S^3\mathscr{E}^*) = 0$ for any $i \geq 2$ and for any $d = 0, 1, 2$.

To prove (i), by the Littlewood-Richardson rule [Man92, Section 2.1.1], we find $S^3\mathscr{E} \otimes S^3\mathscr{E}^* = \mathscr{O}_G \oplus L_{(3,0,\ldots,0,-3)}\mathscr{E}$. Since $n - r \geq 3$ in our setting, the sequence $(0, \ldots, 0, 3, 0, \ldots, 0, -3) - c(n+1)$ is irregular. Hence, $H^q(G, S^3\mathscr{E} \otimes S^3\mathscr{E}^*) = H^q(G, \mathscr{O}_G)$. The latter has dimension 1 when $q = 0$ and 0 otherwise. This proves (i).

To prove (ii), let $L_\lambda\mathscr{E} \subset \bigwedge^i S^3\mathscr{E} \otimes S^3\mathscr{E}^*$ be an irreducible component. Then there is a weight μ such that $L_\lambda\mathscr{E} \subset L_\mu\mathscr{E} \otimes S^3\mathscr{E}^*$. By the Littleword-Richardson rule, for any m, we have $|\lambda|_{>m} \geq |\mu|_{>m} - 3$. Therefore, by Lemma 4.4.16, we find

$$|\lambda|_{>m} \geq i - \binom{m+2}{3} - 3. \tag{4.26}$$

By Theorem 4.4.15, $H^{i-d}(L_\lambda\mathscr{E}) \neq 0$ implies that $(-1,-2,\ldots,-(n-r),\lambda_1-(n-r)-1,\ldots,\lambda_{r+1}-(n+1))$ is regular and that $i-d$ is its inversion number. Equivalently, it means that there exists $0 \leq h \leq r+1$, such that

$$\lambda_h - (n-r) - h \geq 0, \tag{4.27}$$

$$\lambda_{h+1} - (n-r) - (h+1) \leq -(n-r+1), \tag{4.28}$$

$$i - d = h(n-r). \tag{4.29}$$

Here, we use the convention that $\lambda_0 = +\infty$ and that $\lambda_{r+2} = -\infty$ to simplify the notations. We divide into three cases and conclude none of them is possible.

- If $h = r+1$, then by (4.29), $i = (r+1)(n-r)+d > \mathrm{rank} S^3\mathscr{E}$. Hence, $H^{i-d}(\bigwedge^i S^3\mathscr{E} \otimes S^3\mathscr{E}^*) = 0$.
- If $0 < h < r+1$, then we have $i = h(n-r)+d$ by (4.29), and $|\lambda|_{>h} \leq h(r+1-h)$ by (4.28). Combining with (4.26), we obtain

  $$h(n-r)+d-\binom{h+2}{3}-3 \leq h(r+1-h),$$

  or equivalently,

  $$n-2r-1 \leq \frac{1}{6}(h-1)(h-2) + \frac{3-d}{h} =: \phi_d(h). \tag{4.30}$$

  It is not hard to see that

  $$\max_{0<h<r+1} \phi_d(h) \leq \begin{cases} 3-d, & r=2 \\ \frac{1}{6}(r-1)(r-2)+1, & r \geq 3. \end{cases}$$

  A direct comparition with (4.30) gives a contradiction when $r \geq 2$.
- If $h = 0$, then $i = d$ by (4.29). But we have assumed $i \geq 2$, so there is only the case $i = d = 2$ to discuss. In this case, (4.28) shows that $\lambda_1 \leq 0$. We will show that this is impossible. In fact, the irreducible components of $\bigwedge^2 S^3\mathscr{E}$ are $L_{(3,3)}\mathscr{E}$ and $L_{(5,1)}\mathscr{E}$. By the Littlewood-Richardson rule, the λ_1 in $L_{(3,3)}\mathscr{E} \otimes S^3\mathscr{E}^*$ would be ≥ 1 and that in $L_{(5,1)}\mathscr{E} \otimes S^3\mathscr{E}^*$ would be ≥ 3, contradicting the constraints that $\lambda_1 \leq 0$.

□

Proof of Lemma 4.4.14. Let us consider the normal exact sequence

$$0 \to T_X \to T_{G|X} \to N_{X/G} \to 0.$$

Since X is strict Calabi-Yau, $H^0(X,T_X) = 0$. Hence, the kernel of $\beta : H^0(X,N_{X/G}) \to H^1(X,T_X)$ is $H^0(X,T_{G|X})$ and the cokernel is contained in $H^1(X,T_{G|X})$. Hence, to prove

Lemma 4.4.14, it suffices to show that $H^0(X, T_{G|X}) \cong H^0(G, T_G)$ and that $H^1(X, T_{G|X}) = 0$. Considering the short exact sequence

$$0 \to \mathscr{I}_X \otimes T_G \to T_G \to T_{G|X} \to 0,$$

and knowing that $H^1(G, T_G) = 0$, it suffices to show that $H^i(G, \mathscr{I}_X \otimes T_G) = 0$ for $i = 0, 1, 2$. Now we take the Koszul resolution of $\mathscr{I}_X \otimes T_G$, noticing that $T_G = \mathscr{E}^* \otimes \mathscr{Q}$:

$$\ldots \to \bigwedge^{i+1} S^3\mathscr{E} \otimes \mathscr{E}^* \otimes \mathscr{Q} \to \bigwedge^{i} S^3\mathscr{E} \otimes \mathscr{E}^* \otimes \mathscr{Q} \to \ldots \to S^3\mathscr{E} \otimes \mathscr{E}^* \otimes \mathscr{Q} \to \mathscr{I}_X \otimes E^* \otimes \mathscr{Q} \to 0.$$

Therefore, it suffices to show that $H^{i+d}(G, \bigwedge^i S^3\mathscr{E} \otimes \mathscr{E}^* \otimes \mathscr{Q}) = 0$ for any $i \geq 1$ and any $d \in \{-1, 0, 1\}$. To prove this, let $L_\lambda \mathscr{E} \subset \bigwedge^i S^3\mathscr{E} \otimes \mathscr{E}^*$ be an irreducible component. By the Littlewood-Richardson rule and Lemma 4.4.16, we have

$$|\lambda|_{>m} \geq i - \binom{m+2}{3} - 1. \tag{4.31}$$

In order that the sequence $(1, 0, \ldots, 0, \lambda_1, \ldots, \lambda_{r+1}) - c(n+1) = (0, -2, -3, \ldots, -(n-r), \lambda_1 - (n-r) - 1, \ldots, \lambda_{r+1} - (n+1))$ to be regular, and that $H^{i+d}(G, \bigwedge^i S^3\mathscr{E} \otimes \mathscr{E}^* \otimes \mathscr{Q}) \neq 0$, one of the following two cases needs to happen:

(a) There exists $0 \leq h \leq r+1$ such that

$$\lambda_h - (n-r) - h \geq 1, \tag{4.32}$$

$$\lambda_{h+1} - (n-r) - (h+1) \leq -(n-r) - 1, \tag{4.33}$$

$$i + d = h(n-r). \tag{4.34}$$

(b) There exists $0 \leq h \leq r$ such that

$$\lambda_h - (n-r) - h \geq 1, \tag{4.35}$$

$$\lambda_{h+1} - (n-r) - (h+1) = -1, \tag{4.36}$$

$$\lambda_{h+2} - (n-r) - (h+2) \leq -(n-r) - 1, \tag{4.37}$$

$$i + d = h(n-r) + (n-r-1). \tag{4.38}$$

We will show that in both cases, the cohomology is always zero.

For Case (a),

- If $h = r+1$, then (4.34) shows that $i = (r+1)(n-r) - d > \text{rank} S^3\mathscr{E}$, we have $\bigwedge^i S^3\mathscr{E} = 0$.
- If $h = 0$, then (4.34) shows that $i = -d$. Since we suppose $i \geq 1$, only the case $i = -d = 1$ remains to be checked. In this case, (4.33) becomes $\lambda_1 \leq 0$. However, by the Littlewood-Richardson rule, in the decomposition of $S^3\mathscr{E} \otimes \mathscr{E}^*$, $\lambda_1 \geq 2$.
- If $0 < h < r+1$, then $|\lambda|_{>h} \leq h(r+1-h)$ by (4.33). Combining (4.31) and (4.34), we get

  $$h(n-r) - d - \binom{h+2}{3} - 1 \leq h(r+1-h),$$

  and this inequality can be shown to be impossible using similar method as in (4.30).

For Case (b), we have $|\lambda|_{>h} \leq n-r+h+(r-h)(h+1) = n+(r-h)h$. Combining with (4.31) and (4.38), we get

$$h(n-r)+(n-r-1)-1-\binom{h+2}{3}-1 \leq n+(r-h)h,$$

or equivalently,

$$\binom{h+2}{3}-h(h+n-2r)+r+3 \geq 0. \tag{4.39}$$

It is easy to check that (4.39) is impossible if $r=2, h=2$, or if $r \geq 3, 0 < h \leq r$. In fact, define $\phi(h) = \binom{h+2}{3} - h(h+n-2r)+r+3$. Then $\phi(0) = r+3 > 0 \geq \phi(1) = -n+3r+3$. Since $\phi(h)$ is a cubic function, it suffices to show that $\phi(r) < 0$ for $r \geq 2$ and that $\phi(1) < 0$ for $r \geq 3$. The latter is easy to check. For the former, we have

$$\phi(r) = \binom{r+2}{3} - r(n-r)+r+3 = r(-\frac{1}{3}r^2 - r - \frac{5}{3} + \frac{3}{r}) < 0$$

for any $r \geq 2$. The remaining cases that have not been checked are $r=2, h=1$ or $h=0$. To this ends, we consider $|\lambda|_{>h+1}$, by (4.31), (4.37) and (4.38), we get

$$(h+1)(n-r)-3-\binom{h+3}{3} \leq (r-h)(h+1). \tag{4.40}$$

If $r=2, h=1$, then the left-hand-side of (4.40) is 7 whereas the right-hand-side is 2—the inequality (4.40) does not hold. If $h=0$, the left-hand-side of (4.40) is $n-r-4$ whereas the right-hand-side is r—the inequality (4.40) does not hold either. We have eliminated all the possibilities and Lemma 4.4.14 is proven. □

Proof of Proposition 4.4.12

Proof of Proposition 4.4.12. By Hodge symmetry, let us calculate instead the dimension of $H^q(X, \Omega_X)$ for any q. Consider the conormal exact sequence

$$0 \to S^3\mathscr{E}|_X \to (\mathscr{E} \otimes \mathscr{Q}^*)|_X \to \Omega_X \to 0,$$

we need to know the cohomology groups of $S^3\mathscr{E}|_X$ and of $S^3\mathscr{E}|_X$.

Calculation of $H^i(X, S^3\mathscr{E}|_X)$: by the Koszul resolutioin, we need to calculate the cohomology groups of $\bigwedge^k S^3\mathscr{E} \otimes S^3\mathscr{E}$. By the result in Appendix A.2, in particular the output of Cell [5], the only non-zero cohomology groups of the form $H^j(G, \bigwedge^k S^3\mathscr{E} \otimes S^3\mathscr{E})$ are listed as below.

- $h^7(G, \bigwedge^2 S^3\mathscr{E} \otimes S^3\mathscr{E}) = 10.$
- $h^7(G, \bigwedge^3 S^3\mathscr{E} \otimes S^3\mathscr{E}) = 55.$
- $h^{14}(G, \bigwedge^6 S^3\mathscr{E} \otimes S^3\mathscr{E}) = 10.$
- $h^{21}(G, \bigwedge^9 S^3\mathscr{E} \otimes S^3\mathscr{E}) = 1.$
- $h^{21}(G, \bigwedge^{10} S^3\mathscr{E} \otimes S^3\mathscr{E}) = 220.$

The above calculation gives us the following consequences.

- There is an exact sequence

$$0 \to H^4(X, S^3\mathscr{E}|_X) \to H^7(G, \bigwedge^3 S^3\mathscr{E} \otimes S^3\mathscr{E}) \to H^7(G, \bigwedge^2 S^3\mathscr{E} \otimes S^3\mathscr{E}) \to H^5(X, S^3\mathscr{E}|_X) \to 0.$$

  The calculation in Cell [5], together with Theorem 4.4.15, shows that $H^7(G, \bigwedge^3 S^3\mathscr{E} \otimes S^3\mathscr{E}) \cong H^7(G, L_{(10,1,1)}\mathscr{E}) \cong L_{(3,1,\dots,1)}V_{10} \cong S^2V_{10} \otimes \det V_{10}$, that $H^7(G, \bigwedge^2 S^3\mathscr{E} \otimes S^3\mathscr{E}) \cong H^7(G, L_{(8,1)}\mathscr{E}) \cong L_{(1,1,\dots,1,0)}V_{10} \cong V_{10}^* \otimes \det V_{10}$, and that the map $H^7(G, \bigwedge^3 S^3\mathscr{E} \otimes S^3\mathscr{E}) \to H^7(G, \bigwedge^2 S^3\mathscr{E} \otimes S^3\mathscr{E})$ is viewed as $\lrcorner\sigma_f : S^2V_{10} \otimes \det V_{10} \to V_{10}^* \otimes \det V_{10}$, where σ_f is the global section of $H^0(\mathbb{P}^9, \mathscr{O}(3)) \cong S^3H^0(\mathbb{P}^9, \mathscr{O}(1)) = S^3V_{10}^*$, and a generic choice of σ_f gives surjective contraction map

$$\lrcorner\sigma_f : S^2V_{10} \otimes \det V_{10} \to V_{10}^* \otimes \det V_{10}.$$

  Therefore, we conclude that $h^4(X, S^3\mathscr{E}|_X) = 45$ and that $h^5(X, S^3\mathscr{E}|_X) = 0$.

- $h^8(X, S^3\mathscr{E}|_X) = h^{14}(G, \bigwedge^6 S^3\mathscr{E} \otimes S^3\mathscr{E}) = 10$.

- There is an exact sequence

$$0 \to H^{11}(X, S^3\mathscr{E}|_X) \to H^{21}(G, \bigwedge^{10} S^3\mathscr{E} \otimes S^3\mathscr{E}) \to H^{21}(G, \bigwedge^9 S^3\mathscr{E} \otimes S^3\mathscr{E}) \to H^{12}(X, S^3\mathscr{E}|_X) \to 0.$$

  But $H^{12}(X, S^3\mathscr{E}|_X) = 0$ since $\dim X = 11$. Hence, $h^{11}(X, S^3\mathscr{E}|_X) = 219$.

- $H^i(X, S^3\mathscr{E}|_X) = 0$ for other i.

Calculation of $H^i(X, (\mathscr{E} \otimes \mathscr{Q}^*)|_X)$: by the Koszul resolution, we need to calculate the cohomology group of $\bigwedge^k S^3\mathscr{E} \otimes \mathscr{E} \otimes \mathscr{Q}^*$. This is done in Appendix A.2, Cell [6]. By the result there, the only non-zero cohomology groups of the form $H^j(G, \bigwedge^k S^3\mathscr{E} \otimes \mathscr{E} \otimes \mathscr{Q}^*)$ are listed as follows.

- $h^1(G, \bigwedge^0 S^3\mathscr{E} \otimes \mathscr{E} \otimes \mathscr{Q}^*) = 1$.
- $h^{15}(G, \bigwedge^7 S^3\mathscr{E} \otimes \mathscr{E} \otimes \mathscr{Q}^*) = 10$.
- $h^{21}(G, \bigwedge^{10} S^3\mathscr{E} \otimes \mathscr{E} \otimes \mathscr{Q}^*) = 99$.

The above calculation gives us the following consequences.

- $h^1(X, (\mathscr{E} \otimes \mathscr{Q}^*)|_X) = 1$.
- $h^8(X, (\mathscr{E} \otimes \mathscr{Q}^*)|_X) = 10$.
- $h^{11}(X, (\mathscr{E} \otimes \mathscr{Q}^*)|_X) = 99$.
- $h^i(X, (\mathscr{E} \otimes \mathscr{Q}^*)|_X) = 0$ for other i.

Calculation of $H^q(X, \Omega_X)$: We use the conormal exact sequence

$$0 \to S^3\mathscr{E}|_X \to (\mathscr{E} \otimes \mathscr{Q}^*)|_X \to \Omega_X \to 0$$

and the above calculation to obtain:

- $h^1(X, \Omega_X) = 1$.

- $h^3(X,\Omega_X)=45$.
- There is an exact sequence

$$0\to H^7(X,\Omega_X)\to H^8(X,S^3\mathscr{E}|_X)\to H^8(X,(\mathscr{E}\otimes\mathscr{Q}^*)|_X)\to H^8(X,\Omega_X)\to 0.$$

  By chasing the diagram, we find that $H^8(X,S^3\mathscr{E}|_X)\cong H^{14}(G,L_{(10,9,2)}\mathscr{E})\cong L_{(3,1,\dots,1)}V_{10}$ and $H^8(X,\Omega_X)\cong H^{15}(G,L_{(0,\dots,0,-1)}\mathscr{Q}\otimes L_{(10,9,3)}\mathscr{E})\cong L_{(3,1,\dots,1)}V_{10}$ and the map in between is given by the identity map. Therefore, $H^7(X,\Omega_X)=H^8(X,\Omega_X)=0$.
- There is an exact sequence

$$0\to H^{10}(X,\Omega_X)\to H^{11}(X,S^3\mathscr{E}|_X)\to H^{11}(X,(\mathscr{E}\otimes\mathscr{Q}^*)|_X)\to H^{11}(X,\Omega_X)\to 0.$$

  But $h^{11}(X,\Omega_X)\cong h^1(X,K_X)\cong h^{1,0}(X)=0$. Hence, $h^{10}(X,\Omega_X)=219-99=120$. This coincides with our expectation since $h^{10}(X,\Omega_X)=h^1(X,\Omega_X^{10})=h^1(X,T_X)$ and $h^1(X,T_X)=120$ by our calculation in Theorem 4.4.10.
- $h^i(X,\Omega_X)=0$ for other i.

□

Appendix A

Computational Verification in Chapter 4

A.1 Computational Verification in Section 4.3.3

This appendix provides a comprehensive breakdown of the calculations underpinning the results introduced in Section 4.3.3, employing Mathematica and SageMath for computational support.

A.1.1 Computational verification using Mathematica I

(*Define the variables*)

li = Array[*l*, 3]; (*l1, l2, l3*)

mj = Array[*m*, 6]; (*m1, m2, m3, m4, m5, m6*)

(*Create the product expression*)

productExpr = Product[li[[*i*]] + mj[[*j*]], {*i*, 1, 3}, {*j*, 1, 6}];

sym1 = SymmetricReduction[productExpr, mj, {d1, d2, d3, d4, d5, d6}][[1]];

sym2 = SymmetricReduction[sym1, li, {c1, c2, c3}][[1]]

$$c3^6 + c2c3^5d1 + c1c3^5(d1^2 - 2d2) + c2^2c3^4d2 + c1c2c3^4(d1d2 - 3d3) + c2^3c3^3d3 + c3^5(d1^3 - 6d3 + 3(-d1d2 + 3d3)) + c1c2^2c3^3(d1d3 - 4d4) + c2^4c3^2d4 + c1^2c3^4(d2^2 - 6d4 + 2(-d1d3 + 4d4)) + c2c3^4(d1^2d2 - 12d4 + 5(-d1d3 + 4d4) + 2(-d2^2 + 6d4 - 2(-d1d3 + 4d4))) + c1c2^3c3^2(d1d4 - 5d5) + c2^5c3d5 + c1^2c2c3^3(d2d3 - 10d5 + 3(-d1d4 + 5d5)) + c2^2c3^3(d1^2d3 - 20d5 + 7(-d1d4 + 5d5) + 2(-d2d3 + 10d5 - 3(-d1d4 + 5d5))) + c1c3^4(d1d2^2 - 30d5 + 12(-d1d4 + 5d5) + 5(-d2d3 + 10d5 - 3(-d1d4 + 5d5)) + 2(-d1^2d3 + 20d5 - 7(-d1d4 + 5d5) - 2(-d2d3 + 10d5 - 3(-d1d4 + 5d5)))) + c1c2^4c3(d1d5 - 6d6) + c2^6d6 + c1c2^5d1d6 + c1^2c2^4d2d6 + c1^3c2^3d3d6 + c1^4c2^2d4d6 + c1^5c2d5d6 + c1^6d6^2 + c1^5d1d6^2 + c1^4d2d6^2 + c1^3d3d6^2 + c1^2d4d6^2 +$$

$$c1d5d6^2 + d6^3 + c1^2c2^3c3(d2d5 - 5d1d6) + c1^3c2^2c3(d3d5 - 4d2d6) + c2^5(d1^2d6 - 2d2d6) + c1^4c2c3(d4d5 - 3d3d6) + c1c2^4(d1d2d6 - 3d3d6) + c1^2c2^3(d1d3d6 - 4d4d6) + c1^5c3(d5^2 - 2d4d6) + c1^3c2^2(d1d4d6 - 5d5d6) + c1^4c2(d1d5d6 - 6d6^2) + c1^3c2(d2d5d6 - 5d1d6^2) + c1^2c2(d3d5d6 - 4d2d6^2) + c1c2(d4d5d6 - 3d3d6^2) + c2(d5^2d6 - 2d4d6^2) + c1^2c2^2c3^2(d2d4 - 15d6 + 4(-d1d5 + 6d6)) + c2^4c3(d1^2d5 - 11d1d6 + 2(-d2d5 + 5d1d6)) + c1^3c2c3^2(d3d4 - 10d1d6 + 3(-d2d5 + 5d1d6)) + c1^4c3^2(d4^2 - 6d2d6 + 2(-d3d5 + 4d2d6)) + c1c2^3c3(d1d2d5 - 14d2d6 + 5(-d1^2d6 + 2d2d6) + 3(-d3d5 + 4d2d6)) + c1^2c2^2c3(d1d3d5 - 15d3d6 + 4(-d4d5 + 3d3d6) + 4(-d1d2d6 + 3d3d6)) + c2^4(d2^2d6 - 6d4d6 + 2(-d1d3d6 + 4d4d6)) + c1^3c2c3(d1d4d5 - 14d4d6 + 5(-d5^2 + 2d4d6) + 3(-d1d3d6 + 4d4d6)) + c1^4c3(d1d5^2 - 11d5d6 + 2(-d1d4d6 + 5d5d6)) + c1c2^3(d2d3d6 - 10d5d6 + 3(-d1d4d6 + 5d5d6)) + c1^2c2^2(d2d4d6 - 15d6^2 + 4(-d1d5d6 + 6d6^2)) + c1c2^2(d3d4d6 - 10d1d6^2 + 3(-d2d5d6 + 5d1d6^2)) + c2^2(d4^2d6 - 6d2d6^2 + 2(-d3d5d6 + 4d2d6^2)) + c3(d5^3 - 6d3d6^2 + 3(-d4d5d6 + 3d3d6^2)) + c1^3c3^3(d3^2 - 20d6 + 6(-d1d5 + 6d6) + 2(-d2d4 + 15d6 - 4(-d1d5 + 6d6))) + c2^3c3^2(d1^2d4 - 30d6 + 9(-d1d5 + 6d6) + 2(-d2d4 + 15d6 - 4(-d1d5 + 6d6))) + c1c2^2c3^2(d1d2d4 - 35d1d6 + 11(-d2d5 + 5d1d6) + 3(-d3d4 + 10d1d6 - 3(-d2d5 + 5d1d6)) + 4(-d1^2d5 + 11d1d6 - 2(-d2d5 + 5d1d6))) + c1^2c2c3^2(d1d3d4 - 32d2d6 + 10(-d1^2d6 + 2d2d6) + 11(-d3d5 + 4d2d6) + 3(-d1d2d5 + 14d2d6 - 5(-d1^2d6 + 2d2d6) - 3(-d3d5 + 4d2d6)) + 4(-d4^2 + 6d2d6 - 2(-d3d5 + 4d2d6))) + c1^3c3^2(d1d4^2 - 24d3d6 + 9(-d4d5 + 3d3d6) + 6(-d1d2d6 + 3d3d6) + 2(-d1d3d5 + 15d3d6 - 4(-d4d5 + 3d3d6) - 4(-d1d2d6 + 3d3d6))) + c2^3c3(d2^2d5 - 24d3d6 + 6(-d4d5 + 3d3d6) + 9(-d1d2d6 + 3d3d6) + 2(-d1d3d5 + 15d3d6 - 4(-d4d5 + 3d3d6) - 4(-d1d2d6 + 3d3d6))) + c1c2^2c3(d2d3d5 - 32d4d6 + 10(-d5^2 + 2d4d6) + 11(-d1d3d6 + 4d4d6) + 3(-d1d4d5 + 14d4d6 - 5(-d5^2 + 2d4d6) - 3(-d1d3d6 + 4d4d6)) + 4(-d2^2d6 + 6d4d6 - 2(-d1d3d6 + 4d4d6))) + c1^2c2c3(d2d4d5 - 35d5d6 + 11(-d1d4d6 + 5d5d6) + 3(-d2d3d6 + 10d5d6 - 3(-d1d4d6 + 5d5d6)) + 4(-d1d5^2 + 11d5d6 - 2(-d1d4d6 + 5d5d6))) + c2^3(d3^2d6 - 20d6^2 + 6(-d1d5d6 + 6d6^2) + 2(-d2d4d6 + 15d6^2 - 4(-d1d5d6 + 6d6^2))) + c1^3c3(d2d5^2 - 30d6^2 + 9(-d1d5d6 + 6d6^2) + 2(-d2d4d6 + 15d6^2 - 4(-d1d5d6 + 6d6^2))) + c1^2c3(d3d5^2 - 20d1d6^2 + 7(-d2d5d6 + 5d1d6^2) +$$

$$2(-\mathrm{d3d4d6}+10\mathrm{d1d6}^2-3(-\mathrm{d2d5d6}+5\mathrm{d1d6}^2)))+\mathrm{c1c3}(\mathrm{d4d5}^2-12\mathrm{d2d6}^2+5(-\mathrm{d3d5d6}+4\mathrm{d2d6}^2)+2(-\mathrm{d4}^2\mathrm{d6}+6\mathrm{d2d6}^2-2(-\mathrm{d3d5d6}+4\mathrm{d2d6}^2)))+\mathrm{c1c2c3}^3(\mathrm{d1d2d3}-60\mathrm{d6}+22(-\mathrm{d1d5}+6\mathrm{d6})+8(-\mathrm{d2d4}+15\mathrm{d6}-4(-\mathrm{d1d5}+6\mathrm{d6}))+3(-\mathrm{d1}^2\mathrm{d4}+30\mathrm{d6}-9(-\mathrm{d1d5}+6\mathrm{d6})-2(-\mathrm{d2d4}+15\mathrm{d6}-4(-\mathrm{d1d5}+6\mathrm{d6})))+3(-\mathrm{d3}^2+20\mathrm{d6}-6(-\mathrm{d1d5}+6\mathrm{d6})-2(-\mathrm{d2d4}+15\mathrm{d6}-4(-\mathrm{d1d5}+6\mathrm{d6}))))+\mathrm{c1}^2\mathrm{c3}^3(\mathrm{d1d3}^2-50\mathrm{d1d6}+18(-\mathrm{d2d5}+5\mathrm{d1d6})+7(-\mathrm{d3d4}+10\mathrm{d1d6}-3(-\mathrm{d2d5}+5\mathrm{d1d6}))+6(-\mathrm{d1}^2\mathrm{d5}+11\mathrm{d1d6}-2(-\mathrm{d2d5}+5\mathrm{d1d6}))+2(-\mathrm{d1d2d4}+35\mathrm{d1d6}-11(-\mathrm{d2d5}+5\mathrm{d1d6})-3(-\mathrm{d3d4}+10\mathrm{d1d6}-3(-\mathrm{d2d5}+5\mathrm{d1d6}))-4(-\mathrm{d1}^2\mathrm{d5}+11\mathrm{d1d6}-2(-\mathrm{d2d5}+5\mathrm{d1d6}))))+\mathrm{c2}^2\mathrm{c3}^2(\mathrm{d2}^2\mathrm{d4}-53\mathrm{d2d6}+20(-\mathrm{d1}^2\mathrm{d6}+2\mathrm{d2d6})+18(-\mathrm{d3d5}+4\mathrm{d2d6})+7(-\mathrm{d1d2d5}+14\mathrm{d2d6}-5(-\mathrm{d1}^2\mathrm{d6}+2\mathrm{d2d6})-3(-\mathrm{d3d5}+4\mathrm{d2d6}))+6(-\mathrm{d4}^2+6\mathrm{d2d6}-2(-\mathrm{d3d5}+4\mathrm{d2d6}))+2(-\mathrm{d1d3d4}+32\mathrm{d2d6}-10(-\mathrm{d1}^2\mathrm{d6}+2\mathrm{d2d6})-11(-\mathrm{d3d5}+4\mathrm{d2d6})-3(-\mathrm{d1d2d5}+14\mathrm{d2d6}-5(-\mathrm{d1}^2\mathrm{d6}+2\mathrm{d2d6})-3(-\mathrm{d3d5}+4\mathrm{d2d6}))-4(-\mathrm{d4}^2+6\mathrm{d2d6}-2(-\mathrm{d3d5}+4\mathrm{d2d6}))))+\mathrm{c1c2c3}^2(\mathrm{d2d3d4}-60\mathrm{d3d6}+22(-\mathrm{d4d5}+3\mathrm{d3d6})+22(-\mathrm{d1d2d6}+3\mathrm{d3d6})+8(-\mathrm{d1d3d5}+15\mathrm{d3d6}-4(-\mathrm{d4d5}+3\mathrm{d3d6})-4(-\mathrm{d1d2d6}+3\mathrm{d3d6}))+3(-\mathrm{d2}^2\mathrm{d5}+24\mathrm{d3d6}-6(-\mathrm{d4d5}+3\mathrm{d3d6})-9(-\mathrm{d1d2d6}+3\mathrm{d3d6})-2(-\mathrm{d1d3d5}+15\mathrm{d3d6}-4(-\mathrm{d4d5}+3\mathrm{d3d6})-4(-\mathrm{d1d2d6}+3\mathrm{d3d6})))+3(-\mathrm{d1d4}^2+24\mathrm{d3d6}-9(-\mathrm{d4d5}+3\mathrm{d3d6})-6(-\mathrm{d1d2d6}+3\mathrm{d3d6})-2(-\mathrm{d1d3d5}+15\mathrm{d3d6}-4(-\mathrm{d4d5}+3\mathrm{d3d6})-4(-\mathrm{d1d2d6}+3\mathrm{d3d6}))))+\mathrm{c1}^2\mathrm{c3}^2(\mathrm{d2d4}^2-53\mathrm{d4d6}+20(-\mathrm{d5}^2+2\mathrm{d4d6})+18(-\mathrm{d1d3d6}+4\mathrm{d4d6})+7(-\mathrm{d1d4d5}+14\mathrm{d4d6}-5(-\mathrm{d5}^2+2\mathrm{d4d6})-3(-\mathrm{d1d3d6}+4\mathrm{d4d6}))+6(-\mathrm{d2}^2\mathrm{d6}+6\mathrm{d4d6}-2(-\mathrm{d1d3d6}+4\mathrm{d4d6}))+2(-\mathrm{d2d3d5}+32\mathrm{d4d6}-10(-\mathrm{d5}^2+2\mathrm{d4d6})-11(-\mathrm{d1d3d6}+4\mathrm{d4d6})-3(-\mathrm{d1d4d5}+14\mathrm{d4d6}-5(-\mathrm{d5}^2+2\mathrm{d4d6})-3(-\mathrm{d1d3d6}+4\mathrm{d4d6}))-4(-\mathrm{d2}^2\mathrm{d6}+6\mathrm{d4d6}-2(-\mathrm{d1d3d6}+4\mathrm{d4d6}))))+\mathrm{c2}^2\mathrm{c3}(\mathrm{d3}^2\mathrm{d5}-50\mathrm{d5d6}+18(-\mathrm{d1d4d6}+5\mathrm{d5d6})+7(-\mathrm{d2d3d6}+10\mathrm{d5d6}-3(-\mathrm{d1d4d6}+5\mathrm{d5d6}))+6(-\mathrm{d1d5}^2+11\mathrm{d5d6}-2(-\mathrm{d1d4d6}+5\mathrm{d5d6}))+2(-\mathrm{d2d4d5}+35\mathrm{d5d6}-11(-\mathrm{d1d4d6}+5\mathrm{d5d6})-3(-\mathrm{d2d3d6}+10\mathrm{d5d6}-3(-\mathrm{d1d4d6}+5\mathrm{d5d6}))-4(-\mathrm{d1d5}^2+11\mathrm{d5d6}-2(-\mathrm{d1d4d6}+5\mathrm{d5d6}))))+\mathrm{c1c2c3}(\mathrm{d3d4d5}-60\mathrm{d6}^2+22(-\mathrm{d1d5d6}+6\mathrm{d6}^2)+8(-\mathrm{d2d4d6}+15\mathrm{d6}^2-4(-\mathrm{d1d5d6}+6\mathrm{d6}^2))+3(-\mathrm{d2d5}^2+30\mathrm{d6}^2-9(-\mathrm{d1d5d6}+6\mathrm{d6}^2)-2(-\mathrm{d2d4d6}+15\mathrm{d6}^2-4(-\mathrm{d1d5d6}+6\mathrm{d6}^2)))+3(-\mathrm{d3}^2\mathrm{d6}+20\mathrm{d6}^2-6(-\mathrm{d1d5d6}+6\mathrm{d6}^2)-2(-\mathrm{d2d4d6}+15\mathrm{d6}^2-4(-\mathrm{d1d5d6}+6\mathrm{d6}^2))))+\mathrm{c2c3}(\mathrm{d4}^2\mathrm{d5}-30\mathrm{d1d6}^2+12(-\mathrm{d2d5d6}+5\mathrm{d1d6}^2)+5(-\mathrm{d3d4d6}+10\mathrm{d1d6}^2-3(-\mathrm{d2d5d6}+5\mathrm{d1d6}^2))+$$

$$2(-\mathrm{d3d5}^2+20\mathrm{d1d6}^2-7(-\mathrm{d2d5d6}+5\mathrm{d1d6}^2)-2(-\mathrm{d3d4d6}+10\mathrm{d1d6}^2-3(-\mathrm{d2d5d6}+5\mathrm{d1d6}^2))))+\mathrm{c3}^4(\mathrm{d2}^3-90\mathrm{d6}+36(-\mathrm{d1d5}+6\mathrm{d6})+15(-\mathrm{d2d4}+15\mathrm{d6}-4(-\mathrm{d1d5}+6\mathrm{d6}))+6(-\mathrm{d1}^2\mathrm{d4}+30\mathrm{d6}-9(-\mathrm{d1d5}+6\mathrm{d6})-2(-\mathrm{d2d4}+15\mathrm{d6}-4(-\mathrm{d1d5}+6\mathrm{d6})))+6(-\mathrm{d3}^2+20\mathrm{d6}-6(-\mathrm{d1d5}+6\mathrm{d6})-2(-\mathrm{d2d4}+15\mathrm{d6}-4(-\mathrm{d1d5}+6\mathrm{d6})))+3(-\mathrm{d1d2d3}+60\mathrm{d6}-22(-\mathrm{d1d5}+6\mathrm{d6})-8(-\mathrm{d2d4}+15\mathrm{d6}-4(-\mathrm{d1d5}+6\mathrm{d6}))-3(-\mathrm{d1}^2\mathrm{d4}+30\mathrm{d6}-9(-\mathrm{d1d5}+6\mathrm{d6})-2(-\mathrm{d2d4}+15\mathrm{d6}-4(-\mathrm{d1d5}+6\mathrm{d6})))-3(-\mathrm{d3}^2+20\mathrm{d6}-6(-\mathrm{d1d5}+6\mathrm{d6})-2(-\mathrm{d2d4}+15\mathrm{d6}-4(-\mathrm{d1d5}+6\mathrm{d6})))))+\mathrm{c2c3}^3(\mathrm{d2}^2\mathrm{d3}-80\mathrm{d1d6}+31(-\mathrm{d2d5}+5\mathrm{d1d6})+12(-\mathrm{d3d4}+10\mathrm{d1d6}-3(-\mathrm{d2d5}+5\mathrm{d1d6}))+12(-\mathrm{d1}^2\mathrm{d5}+11\mathrm{d1d6}-2(-\mathrm{d2d5}+5\mathrm{d1d6}))+5(-\mathrm{d1d2d4}+35\mathrm{d1d6}-11(-\mathrm{d2d5}+5\mathrm{d1d6})-3(-\mathrm{d3d4}+10\mathrm{d1d6}-3(-\mathrm{d2d5}+5\mathrm{d1d6}))-4(-\mathrm{d1}^2\mathrm{d5}+11\mathrm{d1d6}-2(-\mathrm{d2d5}+5\mathrm{d1d6})))+2(-\mathrm{d1d3}^2+50\mathrm{d1d6}-18(-\mathrm{d2d5}+5\mathrm{d1d6})-7(-\mathrm{d3d4}+10\mathrm{d1d6}-3(-\mathrm{d2d5}+5\mathrm{d1d6}))-6(-\mathrm{d1}^2\mathrm{d5}+11\mathrm{d1d6}-2(-\mathrm{d2d5}+5\mathrm{d1d6}))-2(-\mathrm{d1d2d4}+35\mathrm{d1d6}-11(-\mathrm{d2d5}+5\mathrm{d1d6})-3(-\mathrm{d3d4}+10\mathrm{d1d6}-3(-\mathrm{d2d5}+5\mathrm{d1d6}))-4(-\mathrm{d1}^2\mathrm{d5}+11\mathrm{d1d6}-2(-\mathrm{d2d5}+5\mathrm{d1d6})))))+\mathrm{c1c3}^3(\mathrm{d2d3}^2-80\mathrm{d2d6}+30(-\mathrm{d1}^2\mathrm{d6}+2\mathrm{d2d6})+31(-\mathrm{d3d5}+4\mathrm{d2d6})+12(-\mathrm{d1d2d5}+14\mathrm{d2d6}-5(-\mathrm{d1}^2\mathrm{d6}+2\mathrm{d2d6})-3(-\mathrm{d3d5}+4\mathrm{d2d6}))+12(-\mathrm{d4}^2+6\mathrm{d2d6}-2(-\mathrm{d3d5}+4\mathrm{d2d6}))+5(-\mathrm{d1d3d4}+32\mathrm{d2d6}-10(-\mathrm{d1}^2\mathrm{d6}+2\mathrm{d2d6})-11(-\mathrm{d3d5}+4\mathrm{d2d6})-3(-\mathrm{d1d2d5}+14\mathrm{d2d6}-5(-\mathrm{d1}^2\mathrm{d6}+2\mathrm{d2d6})-3(-\mathrm{d3d5}+4\mathrm{d2d6}))-4(-\mathrm{d4}^2+6\mathrm{d2d6}-2(-\mathrm{d3d5}+4\mathrm{d2d6})))+2(-\mathrm{d2}^2\mathrm{d4}+53\mathrm{d2d6}-20(-\mathrm{d1}^2\mathrm{d6}+2\mathrm{d2d6})-18(-\mathrm{d3d5}+4\mathrm{d2d6})-7(-\mathrm{d1d2d5}+14\mathrm{d2d6}-5(-\mathrm{d1}^2\mathrm{d6}+2\mathrm{d2d6})-3(-\mathrm{d3d5}+4\mathrm{d2d6}))-6(-\mathrm{d4}^2+6\mathrm{d2d6}-2(-\mathrm{d3d5}+4\mathrm{d2d6}))-2(-\mathrm{d1d3d4}+32\mathrm{d2d6}-10(-\mathrm{d1}^2\mathrm{d6}+2\mathrm{d2d6})-11(-\mathrm{d3d5}+4\mathrm{d2d6})-3(-\mathrm{d1d2d5}+14\mathrm{d2d6}-5(-\mathrm{d1}^2\mathrm{d6}+2\mathrm{d2d6})-3(-\mathrm{d3d5}+4\mathrm{d2d6}))-4(-\mathrm{d4}^2+6\mathrm{d2d6}-2(-\mathrm{d3d5}+4\mathrm{d2d6})))))+\mathrm{c3}^3(\mathrm{d3}^3-93\mathrm{d3d6}+36(-\mathrm{d4d5}+3\mathrm{d3d6})+36(-\mathrm{d1d2d6}+3\mathrm{d3d6})+15(-\mathrm{d1d3d5}+15\mathrm{d3d6}-4(-\mathrm{d4d5}+3\mathrm{d3d6})-4(-\mathrm{d1d2d6}+3\mathrm{d3d6}))+6(-\mathrm{d2}^2\mathrm{d5}+24\mathrm{d3d6}-6(-\mathrm{d4d5}+3\mathrm{d3d6})-9(-\mathrm{d1d2d6}+3\mathrm{d3d6})-2(-\mathrm{d1d3d5}+15\mathrm{d3d6}-4(-\mathrm{d4d5}+3\mathrm{d3d6})-4(-\mathrm{d1d2d6}+3\mathrm{d3d6})))+6(-\mathrm{d1d4}^2+24\mathrm{d3d6}-9(-\mathrm{d4d5}+3\mathrm{d3d6})-6(-\mathrm{d1d2d6}+3\mathrm{d3d6})-2(-\mathrm{d1d3d5}+15\mathrm{d3d6}-4(-\mathrm{d4d5}+3\mathrm{d3d6})-4(-\mathrm{d1d2d6}+3\mathrm{d3d6})))+3(-\mathrm{d2d3d4}+60\mathrm{d3d6}-22(-\mathrm{d4d5}+3\mathrm{d3d6})-22(-\mathrm{d1d2d6}+3\mathrm{d3d6})-8(-\mathrm{d1d3d5}+15\mathrm{d3d6}-4(-\mathrm{d4d5}+3\mathrm{d3d6})-4(-\mathrm{d1d2d6}+3\mathrm{d3d6}))-3(-\mathrm{d2}^2\mathrm{d5}+24\mathrm{d3d6}-6(-\mathrm{d4d5}+3\mathrm{d3d6})-9(-\mathrm{d1d2d6}+3\mathrm{d3d6})-2(-\mathrm{d1d3d5}+15\mathrm{d3d6}-4(-\mathrm{d4d5}+3\mathrm{d3d6})-4(-\mathrm{d1d2d6}+$$

3d3d6))) − 3(−d1d4^2 + 24d3d6 − 9(−d4d5 + 3d3d6) − 6(−d1d2d6 + 3d3d6) − 2(−d1d3d5 + 15d3d6 − 4(−d4d5 + 3d3d6) − 4(−d1d2d6 + 3d3d6))))) + c2c3^2(d3^{2}d4 − 80d4d6 + 30(−d5^2 + 2d4d6) + 31(−d1d3d6 + 4d4d6) + 12(−d1d4d5 + 14d4d6 − 5(−d5^2 + 2d4d6) − 3(−d1d3d6 + 4d4d6)) + 12(−d2^{2}d6 + 6d4d6 − 2(−d1d3d6 + 4d4d6)) + 5(−d2d3d5 + 32d4d6 − 10(−d5^2 + 2d4d6) − 11(−d1d3d6 + 4d4d6) − 3(−d1d4d5 + 14d4d6 − 5(−d5^2 + 2d4d6) − 3(−d1d3d6 + 4d4d6)) − 4(−d2^{2}d6 + 6d4d6 − 2(−d1d3d6 + 4d4d6))) + 2(−d2d4^2 + 53d4d6 − 20(−d5^2 + 2d4d6) − 18(−d1d3d6 + 4d4d6) − 7(−d1d4d5 + 14d4d6 − 5(−d5^2 + 2d4d6) − 3(−d1d3d6 + 4d4d6)) − 6(−d2^{2}d6 + 6d4d6 − 2(−d1d3d6 + 4d4d6)) − 2(−d2d3d5 + 32d4d6 − 10(−d5^2 + 2d4d6) − 11(−d1d3d6 + 4d4d6) − 3(−d1d4d5 + 14d4d6 − 5(−d5^2 + 2d4d6) − 3(−d1d3d6 + 4d4d6)) − 4(−d2^{2}d6 + 6d4d6 − 2(−d1d3d6 + 4d4d6))))) + c1c3^2(d3d4^2 − 80d5d6 + 31(−d1d4d6 + 5d5d6) + 12(−d2d3d6 + 10d5d6 − 3(−d1d4d6 + 5d5d6)) + 12(−d1d5^2 + 11d5d6 − 2(−d1d4d6 + 5d5d6)) + 5(−d2d4d5 + 35d5d6 − 11(−d1d4d6 + 5d5d6) − 3(−d2d3d6 + 10d5d6 − 3(−d1d4d6 + 5d5d6)) − 4(−d1d5^2 + 11d5d6 − 2(−d1d4d6 + 5d5d6))) + 2(−d3^{2}d5 + 50d5d6 − 18(−d1d4d6 + 5d5d6) − 7(−d2d3d6 + 10d5d6 − 3(−d1d4d6 + 5d5d6)) − 6(−d1d5^2 + 11d5d6 − 2(−d1d4d6 + 5d5d6)) − 2(−d2d4d5 + 35d5d6 − 11(−d1d4d6 + 5d5d6) − 3(−d2d3d6 + 10d5d6 − 3(−d1d4d6 + 5d5d6)) − 4(−d1d5^2 + 11d5d6 − 2(−d1d4d6 + 5d5d6))))) + c3^2(d4^3 − 90d6^2 + 36(−d1d5d6 + 6d6^2) + 15(−d2d4d6 + 15d6^2 − 4(−d1d5d6 + 6d6^2)) + 6(−d2d5^2 + 30d6^2 − 9(−d1d5d6 + 6d6^2) − 2(−d2d4d6 + 15d6^2 − 4(−d1d5d6 + 6d6^2))) + 6(−d3^{2}d6 + 20d6^2 − 6(−d1d5d6 + 6d6^2) − 2(−d2d4d6 + 15d6^2 − 4(−d1d5d6 + 6d6^2))) + 3(−d3d4d5 + 60d6^2 − 22(−d1d5d6 + 6d6^2) − 8(−d2d4d6 + 15d6^2 − 4(−d1d5d6 + 6d6^2)) − 3(−d2d5^2 + 30d6^2 − 9(−d1d5d6 + 6d6^2) − 2(−d2d4d6 + 15d6^2 − 4(−d1d5d6 + 6d6^2))) − 3(−d3^{2}d6 + 20d6^2 − 6(−d1d5d6 + 6d6^2) − 2(−d2d4d6 + 15d6^2 − 4(−d1d5d6 + 6d6^2)))))

A.1.2 Computational verification using Mathematica II

(*Define the Chern classes of the symmetric product*)

```
S1[c1_,c2_,c3_]:=4c1;
S2[c1_,c2_,c3_]:=5c1^2+5c2;
S3[c1_,c2_,c3_]:=2c1^3+11c1c2+7c3;

(*Define the Euler classes up to degree 3 in ci*)

e1 = d6^3+c1d5d6^2+c1^2d4d6^2+c2(d5^2d6-2d4d6^2)+c1c2(d4d5d6-3d3d6^2)
+c3(d5^3-6d3d6^2+3(-d4d5d6+3d3d6^2))+c1^3d3d6^2;
e2 = (C1-c1)^6+(C1-c1)^5S1[c1,c2,c3]+(C1-c1)^4S2[c1,c2,c3]+
(C1-c1)^3S3[c1,c2,c3];
e3 = 8(C1-c1)^3+4(C1-c1)^2c1+2(C1-c1)c2+c3;

(*The class of [Tilde F] up to degree 3*)

final = e1e2e3;
final = Expand[final];
Collect[final,{c1,c2,c3}]
```

$$8C1^9d6^3 + c3^3(7C1^3d5^3 - 21C1^3d4d5d6 + 21C1^3d3d6^2) + c2^3(10C1^5d5^2d6 - 20C1^5d4d6^2) + c1^9(4C1^2d4d6^2 + 24c2d4d6^2) + c3^2(57C1^6d5^3 - 171C1^6d4d5d6 + 171C1^6d3d6^2 + 7C1^3d6^3) + c3(8C1^9d5^3 - 24C1^9d4d5d6 + 24C1^9d3d6^2 + 57C1^6d6^3) + c2^2(42C1^7d5^2d6 - 84C1^7d4d6^2 + 10C1^5d6^3 + c3(10C1^5d5^3 - 30C1^5d4d5d6 + 19C1^4d5^2d6 + 30C1^5d3d6^2 - 38C1^4d4d6^2)) + c1^8(-24C1^3d4d6^2 + 28c3d4d6^2 + 4C1^2d5d6^2 + c2^2(24d4d5d6 - 72d3d6^2) + c2(4C1^2d4d5d6 - 12C1^2d3d6^2 - 148C1d4d6^2 + 24d5d6^2)) + c2(8C1^9d5^2d6 - 16C1^9d4d6^2 + 42C1^7d6^3 + c3^2(19C1^4d5^3 - 57C1^4d4d5d6 + 7C1^3d5^2d6 + 57C1^4d3d6^2 - 14C1^3d4d6^2) + c3(42C1^7d5^3 - 126C1^7d4d5d6 + 57C1^6d5^2d6 + 126C1^7d3d6^2 - 114C1^6d4d6^2 + 19C1^4d6^3)) + c1^7(52C1^4d4d6^2 - 24C1^3d5d6^2 + 4C1^2d6^3 + c2^2(-148C1d4d5d6 + 24d5^2d6 + 444C1d3d6^2 - 36d4d6^2) + c3(4C1^2d5^3 - 12C1^2d4d5d6 + 12C1^2d3d6^2 - 196C1d4d6^2 + 28d5d6^2) + c2(-24C1^3d4d5d6 + 4C1^2d5^2d6 + 72C1^3d3d6^2 + 334C1^2d4d6^2 - 148C1d5d6^2 +$$

$24d6^3 + c3(24d5^3 - 44d4d5d6 - 12d3d6^2))) + c1^6(-40C1^5d4d6^2 + 52C1^4d5d6^2 - 24C1^3d6^3 + c2^3(12d4d5d6 - 36d3d6^2) + c3^2(28d5^3 - 84d4d5d6 + 84d3d6^2) + c2^2(342C1^2d4d5d6 - 148C1d5^2d6 - 1026C1^2d3d6^2 + 258C1d4d6^2 + 12d5d6^2) + c3(-24C1^3d5^3 + 72C1^3d4d5d6 - 72C1^3d3d6^2 + 559C1^2d4d6^2 - 196C1d5d6^2 + 28d6^3) + c2(52C1^4d4d5d6 - 24C1^3d5^2d6 - 156C1^4d3d6^2 - 278C1^3d4d6^2 + 342C1^2d5d6^2 - 148C1d6^3 + c3(-148C1d5^3 + 248C1d4d5d6 + 28d5^2d6 + 144C1d3d6^2 - 48d4d6^2))) + c1^5(-20C1^6d4d6^2 - 40C1^5d5d6^2 + 52C1^4d6^3 + c2^3(-38C1d4d5d6 + 12d5^2d6 + 114C1d3d6^2 - 24d4d6^2) + c3^2(-196C1d5^3 + 588C1d4d5d6 - 588C1d3d6^2 - 7d4d6^2) + c3(52C1^4d5^3 - 156C1^4d4d5d6 + 156C1^4d3d6^2 - 838C1^3d4d6^2 + 559C1^2d5d6^2 - 196C1d6^3) + c2^2(-326C1^3d4d5d6 + 342C1^2d5^2d6 + 978C1^3d3d6^2 - 652C1^2d4d6^2 - 38C1d5d6^2 + 12d6^3 + c3(12d5^3 - 28d4d5d6 + 12d3d6^2)) + c2(-40C1^5d4d5d6 + 52C1^4d5^2d6 + 120C1^5d3d6^2 - 100C1^4d4d6^2 - 326C1^3d5d6^2 + 342C1^2d6^3 + c3(342C1^2d5^3 - 467C1^2d4d5d6 - 196C1d5^2d6 - 651C1^2d3d6^2 + 349C1d4d6^2 + 8d5d6^2))) + c1^4(56C1^7d4d6^2 - 20C1^6d5d6^2 - 40C1^5d6^3 + c2^3(32C1^2d4d5d6 - 38C1d5^2d6 - 96C1^2d3d6^2 + 76C1d4d6^2) + c3^2(559C1^2d5^3 - 1677C1^2d4d5d6 + 1677C1^2d3d6^2 + 21C1d4d6^2 - 7d5d6^2) + c3(-40C1^5d5^3 + 120C1^5d4d5d6 - 120C1^5d3d6^2 + 700C1^4d4d6^2 - 838C1^3d5d6^2 + 559C1^2d6^3) + c2^2(4C1^4d4d5d6 - 326C1^3d5^2d6 - 12C1^4d3d6^2 + 664C1^3d4d6^2 + 32C1^2d5d6^2 - 38C1d6^3 + c3(-38C1d5^3 + 71C1d4d5d6 + 8d5^2d6 + 15C1d3d6^2 - 16d4d6^2)) + c2(-20C1^6d4d5d6 - 40C1^5d5^2d6 + 60C1^6d3d6^2 + 320C1^5d4d6^2 + 4C1^4d5d6^2 - 326C1^3d6^3 + c3^2(8d5^3 - 31d4d5d6 + 45d3d6^2) + c3(-326C1^3d5^3 + 140C1^3d4d5d6 + 559C1^2d5^2d6 + 1536C1^3d3d6^2 - 1037C1^2d4d6^2 - 43C1d5d6^2 + 8d6^3))) + c1^3(8C1^9d3d6^2 - 36C1^8d4d6^2 + 56C1^7d5d6^2 - 20C1^6d6^3 + c3^3(-7d5^3 + 21d4d5d6 - 21d3d6^2) + c2^3(12C1^3d4d5d6 + 32C1^2d5^2d6 - 36C1^3d3d6^2 - 64C1^2d4d6^2) + c3^2(-838C1^3d5^3 + 2514C1^3d4d5d6 - 2514C1^3d3d6^2 - 21C1^2d4d6^2 + 21C1d5d6^2 - 7d6^3) + c3(-20C1^6d5^3 + 60C1^6d4d5d6 - 60C1^6d3d6^2 - 310C1^5d4d6^2 + 700C1^4d5d6^2 - 838C1^3d6^3) + c2^2(240C1^5d4d5d6 + 4C1^4d5^2d6 - 720C1^5d3d6^2 - 36C1^4d4d6^2 + 12C1^3d5d6^2 + 32C1^2d6^3 + c3(32C1^2d5^3 - 15C1^2d4d5d6 - 43C1d5^2d6 - 147C1^2d3d6^2 + 86C1d4d6^2)) + c2(56C1^7d4d5d6 - 20C1^6d5^2d6 - 168C1^7d3d6^2 - 138C1^6d4d6^2 + 240C1^5d5d6^2 + 4C1^4d6^3 + c3^2(-43C1d5^3 +$

$150C1d4d5d6 - 7d5^2d6 - 192C1d3d6^2 + 14d4d6^2) + c3(4C1^4d5^3 + 688C1^4d4d5d6 - 838C1^3d5^2d6 - 2088C1^4d3d6^2 + 1611C1^3d4d6^2 + 81C1^2d5d6^2 - 43C1d6^3))) + c1^2(8C1^9d4d6^2 - 36C1^8d5d6^2 + 56C1^7d6^3 + c3^3(21C1d5^3 - 63C1d4d5d6 + 63C1d3d6^2) + c2^3(-28C1^4d4d5d6 + 12C1^3d5^2d6 + 84C1^4d3d6^2 - 24C1^3d4d6^2) + c3^2(700C1^4d5^3 - 2100C1^4d4d5d6 + 2100C1^4d3d6^2 + 7C1^3d4d6^2 - 21C1^2d5d6^2 + 21C1d6^3) + c3(56C1^7d5^3 - 168C1^7d4d5d6 + 168C1^7d3d6^2 + 57C1^6d4d6^2 - 310C1^5d5d6^2 + 700C1^4d6^3) + c2^2(-178C1^6d4d5d6 + 240C1^5d5^2d6 + 534C1^6d3d6^2 - 470C1^5d4d6^2 - 28C1^4d5d6^2 + 12C1^3d6^3 + c3(12C1^3d5^3 - 101C1^3d4d5d6 + 81C1^2d5^2d6 + 231C1^3d3d6^2 - 162C1^2d4d6^2)) + c2(-36C1^8d4d5d6 + 56C1^7d5^2d6 + 108C1^8d3d6^2 - 70C1^7d4d6^2 - 178C1^6d5d6^2 + 240C1^5d6^3 + c3^2(81C1^2d5^3 - 264C1^2d4d5d6 + 21C1d5^2d6 + 306C1^2d3d6^2 - 42C1d4d6^2) + c3(240C1^5d5^3 - 1030C1^5d4d5d6 + 700C1^4d5^2d6 + 1650C1^5d3d6^2 - 1381C1^4d4d6^2 - 65C1^3d5d6^2 + 81C1^2d6^3))) + c1(8C1^9d5d6^2 - 36C1^8d6^3 + c3^3(-21C1^2d5^3 + 63C1^2d4d5d6 - 63C1^2d3d6^2) + c2^3(10C1^5d4d5d6 - 28C1^4d5^2d6 - 30C1^5d3d6^2 + 56C1^4d4d6^2) + c3^2(-310C1^5d5^3 + 930C1^5d4d5d6 - 930C1^5d3d6^2 + 7C1^3d5d6^2 - 21C1^2d6^3) + c3(-36C1^8d5^3 + 108C1^8d4d5d6 - 108C1^8d3d6^2 + 57C1^6d5d6^2 - 310C1^5d6^3) + c2^2(42C1^7d4d5d6 - 178C1^6d5^2d6 - 126C1^7d3d6^2 + 356C1^6d4d6^2 + 10C1^5d5d6^2 - 28C1^4d6^3 + c3(-28C1^4d5^3 + 103C1^4d4d5d6 - 65C1^3d5^2d6 - 141C1^4d3d6^2 + 130C1^3d4d6^2)) + c2(8C1^9d4d5d6 - 36C1^8d5^2d6 - 24C1^9d3d6^2 + 72C1^8d4d6^2 + 42C1^7d5d6^2 - 178C1^6d6^3 + c3^2(-65C1^3d5^3 + 202C1^3d4d5d6 - 21C1^2d5^2d6 - 216C1^3d3d6^2 + 42C1^2d4d6^2) + c3(-178C1^6d5^3 + 591C1^6d4d5d6 - 310C1^5d5^2d6 - 705C1^6d3d6^2 + 620C1^5d4d6^2 + 19C1^4d5d6^2 - 65C1^3d6^3)))$

A.1.3 Computational verification using SageMath I

```
In [1]: # Define the Schur's polynomials
        R = SymmetricFunctions(QQ)
        s = R.schur()

        #The expressions of the coefficients
        coef_of_c1c1c1 = -36 * s[1]^8 * s[4] * s[6]^2 + \
            56 * s[1]^7 * s[5] * s[6]^2 - 20 * s[1]^6 * s[6]^3 + \
            8 * s[1]^9 * s[3] * s[6]^2
```

```
        coef_of_c1c2 = 8 * s[1]^9 * s[4] * s[5] * s[6] - \
            36 * s[1]^8 * s[5]^2 * s[6] - 24 * s[1]^9 * s[3] * s[6]^2 + \
            72 * s[1]^8 * s[4] * s[6]^2 + 42 * s[1]^7 * s[5] * s[6]^2 - \
            178 * s[1]^6 * s[6]^3
        coef_of_c3 = 8 * s[1]^9 * s[5]^3 - \
            24 * s[1]^9 * s[4] * s[5] * s[6] + \
            24 * s[1]^9 * s[3] * s[6]^2 + 57 * s[1]^6 * s[6]^3

        #Define a function that finds the coefficient of s[6,6,6,6]
        def find_num(expr):
            for part in expr:
                if part[0]==[6,6,6,6]:
                    return part[1]

        #Print the final result
        print(
            f'The expression of [F] is ({find_num(coef_of_c1c1c1)}).c1^3 \
        + ({find_num(coef_of_c1c2)}).c1c2 + ({find_num(coef_of_c3)}).c3.'
        )
```

```
The expression of [F] is (-20).c1^3 + (110).c1c2 + (49).c3.
```

A.2 Computational Verification in Section 4.4.3

```
In [1]: R = SymmetricFunctions(QQ)
        s = R.schur()
```

```
In [2]: def mergeSortAndCount(arr, temp_arr, left, right):
            # Initialize the inversion count
            inv_count = 0

            # If the list is not yet of length one
            if left < right:
                # Mid is the point where the array is divided into two subarrays
                mid = (left + right) // 2

                # Inversion count will be the sum of inversions in left-part, \
                # right-part and number of inversions in merging
                inv_count += mergeSortAndCount(arr, temp_arr, left, mid)
                inv_count += mergeSortAndCount(arr, temp_arr, mid + 1, right)

                # Merging the two parts
                inv_count += merge(arr, temp_arr, left, mid, right)

            return inv_count
```

```
        def merge(arr, temp_arr, left, mid, right):
            i = left      # Starting index of left subarray
            j = mid + 1   # Starting index of right subarray
            k = left      # Starting index of to be sorted subarray
            inv_count = 0

            # Conditions are checked to make sure that i and j don't \
            # exceed their subarray limits
            while i <= mid and j <= right:
                # There will be no inversion if arr[i] <= arr[j]
                if arr[i] <= arr[j]:
                    temp_arr[k] = arr[i]
                    k += 1
                    i += 1
                else:
                    # Inversion will occur
                    temp_arr[k] = arr[j]
                    inv_count += (mid - i + 1)  # Count the inversions
                    k += 1
                    j += 1

            # Copy the remaining elements of left subarray into temporary array
            while i <= mid:
                temp_arr[k] = arr[i]
                k += 1
                i += 1

            # Copy the remaining elements of right subarray into temporary array
            while j <= right:
                temp_arr[k] = arr[j]
                k += 1
                j += 1

            # Copy the sorted subarray into Original array
            for loop_var in range(left, right + 1):
                arr[loop_var] = temp_arr[loop_var]

            return inv_count

        def inversionCount(arr):
            # Temporary array to be used in merge function
            temp_arr = [0]*len(arr)
            return mergeSortAndCount(arr, temp_arr, 0, len(arr) - 1)
```

```
In [3]: def dimension(w, n):
            # The dimension of the representation L_wV_n.

            R=SymmetricFunctions(QQ)
            if w[-1] < 0:
```

```
            w = [-w[-i] for i in range(1, len(w)+1)]
        return sum(s[w].expand(n).coefficients())
```

```
In [4]: def calculate_cohomology(schur_e, schur_q, r, n):

            """
            Given:
                the Schur weights of the tautological sub and quotient \
                bundles over the Grassmannian Gr(r, n),
            return:
                None in the irregular case;
                the index and the dimension of the cohomology in the \
                regualar case.
            """

            # Extend schur_e and schur_q lists with zeros
            schur_e += [0] * (r - len(schur_e))
            schur_q += [0] * (n - r - len(schur_q))

            # Concatenate schur_e and schur_q
            schur = schur_q + schur_e

            # Generate delta list
            delta = [i for i in range(1, n+1)]



            # Calculate N
            N = r * (n - r)


            # Element-wise subtraction of delta from schur
            schur_delta = [s - d for s, d in zip(schur, delta)]

            # Check for no duplicates and non-negative values
            if len(schur_delta) == len(set(schur_delta)):
                ind = inversionCount(schur_delta[::-1])
                schur_delta_sorted = sorted(schur_delta, reverse=True)
                weight = [s + d for s, d in zip(schur_delta_sorted, delta)]
                return (ind, dimension(tuple(weight), n))
            else:
                return None
```

```
In [5]: R = SymmetricFunctions(QQ)
        s = R.schur()
        r = 3 #dealing with planes
        d = binomial(r+2,3) #d is the rank of Sym^3E
        for i in range(d+1):
```

```
            # In the following code, we print the Schur decomposition of \
            # Wedge^iSym^3E\otimes Sym^3E
            schur = s([1]*i)(s[3]) * s[3]
            part_list = list()
            coh_dict = dict()
            for part in schur:
                if len(part[0]) <= r:
                    # print(f"Working on {part[0]}...")
                    part_list.append(part)
                    coh = calculate_cohomology(part[0], [0], 3, 10)
                    if coh:
                        if coh[0] in coh_dict:
                            coh_dict[coh[0]] += part[1] * coh[1]
                        else:
                            coh_dict[coh[0]] = part[1] * coh[1]
                        print(f"{part[0]} is regular: H^{coh[0]} = {coh[1]}")
                    # else:
                        # print(" irregular")

            # print(f"The decomp of Wedge^{i}Sym^3E otimes Sym^3E is {part_list}"
            for ind in coh_dict:
                print(f"The dimension of H^{ind}(Wedge^{i}Sym^3E otimes Sym^3E) \
        is {coh_dict[ind]}")
            if coh_dict:
                print(f"And all the other cohomology of Wedge^{i}Sym^3E otimes \
        Sym^3E is 0")
            else:
                print(f"All the cohomology of Wedge^{i}Sym^3E otimes Sym^3E is 0"
```

```
All the cohomology of Wedge^0Sym^3E otimes Sym^3E is 0
All the cohomology of Wedge^1Sym^3E otimes Sym^3E is 0
[8, 1] is regular: H^7 = 10
The dimension of H^7(Wedge^2Sym^3E otimes Sym^3E) is 10
And all the other cohomology of Wedge^2Sym^3E otimes Sym^3E is 0
[10, 1, 1] is regular: H^7 = 55
The dimension of H^7(Wedge^3Sym^3E otimes Sym^3E) is 55
And all the other cohomology of Wedge^3Sym^3E otimes Sym^3E is 0
All the cohomology of Wedge^4Sym^3E otimes Sym^3E is 0
All the cohomology of Wedge^5Sym^3E otimes Sym^3E is 0
[10, 9, 2] is regular: H^14 = 10
The dimension of H^14(Wedge^6Sym^3E otimes Sym^3E) is 10
And all the other cohomology of Wedge^6Sym^3E otimes Sym^3E is 0
All the cohomology of Wedge^7Sym^3E otimes Sym^3E is 0
All the cohomology of Wedge^8Sym^3E otimes Sym^3E is 0
[10, 10, 10] is regular: H^21 = 1
```

```
The dimension of H^21(Wedge^9Sym^3E otimes Sym^3E) is 1
And all the other cohomology of Wedge^9Sym^3E otimes Sym^3E is 0
[13, 10, 10] is regular: H^21 = 220
The dimension of H^21(Wedge^10Sym^3E otimes Sym^3E) is 220
And all the other cohomology of Wedge^10Sym^3E otimes Sym^3E is 0
```

```
In [6]: R = SymmetricFunctions(QQ)
        s = R.schur()
        r = 2 + 1 #dealing with planes
        n = (r + 2) * (r + 1) // 2
        d = binomial(r+2,3) #d is the rank of Sym^3E
        for i in range(d+1):
            # In the following code, we print the Schur decomposition \
            # of Wedge^iSym^3E\otimes E\otimes Q^*
            schur = s([1]*i)(s[3]) * s[1]
            part_list = list()
            coh_dict = dict()
            for part in schur:
                if len(part[0]) <= r:
                    # print(f"Working on {part[0]}...")
                    part_list.append(part)
                    coh = calculate_cohomology(part[0], [0]*(n-r-1)+[-1], \
                                               r, n)
                    if coh:
                        if coh[0] in coh_dict:
                            coh_dict[coh[0]] += part[1] * coh[1]
                        else:
                            coh_dict[coh[0]] = part[1] * coh[1]
                        print(f"{part[0]} is regular: H^{coh[0]} = {coh[1]}")
                    # else:
                        # print(" irregular")

            # print(f"The decomp of Wedge^{i}Sym^3E otimes E otimes Q^* is \
            # {part_list}")
            for ind in coh_dict:
                print(f"The dimension of H^{ind}(Wedge^{i}Sym^3E otimes E \
        otimes Q^*) is {coh_dict[ind]}")
            if coh_dict:
                print(f"And all the other cohomology of Wedge^{i}Sym^3E \
        otimes E otimes Q^* is 0")
            else:
                print(f"All the cohomology of Wedge^{i}Sym^3E otimes E \
        otimes Q^* is 0")
```

```
[1] is regular: H^1 = 1
The dimension of H^1(Wedge^0Sym^3E otimes E otimes Q^*) is 1
And all the other cohomology of Wedge^0Sym^3E otimes E otimes Q^* is 0
```

```
All the cohomology of Wedge^1Sym^3E otimes E otimes Q^* is 0
All the cohomology of Wedge^2Sym^3E otimes E otimes Q^* is 0
All the cohomology of Wedge^3Sym^3E otimes E otimes Q^* is 0
All the cohomology of Wedge^4Sym^3E otimes E otimes Q^* is 0
All the cohomology of Wedge^5Sym^3E otimes E otimes Q^* is 0
All the cohomology of Wedge^6Sym^3E otimes E otimes Q^* is 0
[10, 9, 3] is regular: H^15 = 10
The dimension of H^15(Wedge^7Sym^3E otimes E otimes Q^*) is 10
And all the other cohomology of Wedge^7Sym^3E otimes E otimes Q^* is 0
```

Bibliography

[AP92] A. Alzati and G. P. Pirola, On the holomorphic length of a complex projective variety, Arch. Math. (Basel) 59 (1992), no. 4, 398–402. MR 1179468.

[Ame09] E. Amerik, A computation of invariants of a rational self-map, Ann. Fac. Sci. Toulouse Math. 18 (2009), no. 3, 445–457

[ABR11] E. Amerik, F. Bogomolov, and M. Rovinsky, *Remarks on Endomorphisms and Rational Points,* Compositio Mathematica 147, no. 6 (2011): 1819–42.

[AC08] E. Amerik and F. Campana, *Pure and Applied Mathematics Quarterly*, Volume 4, Number 2 (Special Issue: In honor of Fedor Bogomolov, Part 1 of 2) 509—545, 2008.

[AV08] E. Amerik, C. Voisin, *Potential density of rational points on the variety of lines of a cubic fourfold*, Duke Math. Journal, Vol 145, n0 2, 379-408, (2008).

[ACGH85] E. Arbarello, M. Cornalba, P. A. Griffiths, J. Harris, *Geometry of Algebraic Curves, Volume I*, 1985, Springer.

[Bai23] C. Bai, *On Abel–Jacobi maps of Lagrangian families,* Math. Z. 304, 34 (2023).

[Bai24] C. Bai, *On some birational invariants of hyper-Kähler manifolds*, International Mathematics Research Notices, Volume 2024, Issue 10, May 2024, Pages 8628–8637.

[Bai24II] C. Bai, *On the geometry of the higher dimensional Voisin maps*, ArXiv: 2404.10138.

[BDELU17] F. Bastianelli, P. De Poi, L. Ein, R. Lazarsfeld, B. Ullery, *Measures of irrationality for hypersurfaces of large degree*, Compos. Math. 153 (2017), no. 11, 2368-2393.

[BCD] Bastianelli, Cortini, and De Poi, *The gonality theorem of Noether for hypersurfaces*, J. Algebraic Geom., 23 (2014), no. 2, 313–339.

[Baz17] I. Bazhov, *On the Chow group of zero-cycles of Calabi-Yau hypersurfaces*, manuscripta mathematica 152(1), 2017.

[Bea83] A. Beauville, *Variétés kählériennes dont la première classe de Chern est nulle*, J. Differential Geometry 18(1983), 755-782.

[Bea07] A. Beauville. On the splitting of the Bloch-Beilinson filtration, in Algebraic cycles and motives (vol. 2), London Math. Soc. Lecture Notes 344, 38-53; Cambridge University Press (2007).

[BD85] A. Beauville, R. Donagi, *The variety of lines of a cubic fourfold*, C. R. Acad. Sci. Paris 301 Séries I, (1985), 703-706.

[Blo80] S. Bloch, *Lectures on algebraic cycles,* Duke Univ. Math. Ser. IV (1980).

[BS83] S. Bloch, and V. Srinivas, *Remarks on correspondences and algebraic cycles,* Amer. J. Math., 105(5):1235–1253, 1983.

[Bot75] R. Bott, Homogeneous vector bundles, Ann. of Math. (2) 66 (1975), 203–248.

[Bri77] J. Briançon. *Description de $Hilb^n\mathbb{C}\{x,y\}$*, Invent. Math., 41:45–89, 1977.

[CG72] H. Clemens, P. Griffiths, *The intermediate Jacobian of the cubic threefold*, Ann. of Math. (2) 95 (1972), 281–356.

[DM98] O. Debarre, L. Manivel, *Sur la variété des espaces linéaires contenus dans une intersection complète*, Math. Ann. 312 (1998), 549-574.

[dCM02] M. A. de Cataldo, L. Migliorini, *The Chow groups and the motive of the Hilbert scheme of points on a surface*, J. algebra 251 (2002), 824-848.

[Del71] P. Deligne, *La conjecture de Weil pour les surfaces K3*, Inventiones mathematicae, 15(3):206–226, 1971.

[Dem76] B. Demazure, *A very simple proof of Bott's theorem*, Invent. Math. 33, 271-272, (1976).

[DM98] R. Donagi, E. Markman, *Spectral covers, algebraically completely integrable, Hamiltonian systems, and moduli of bundles*, Integrable Systems and Quantum Groups (Montecatini Terme 1993), Lecture Notes in Math., 1620, pp. 1-119. Springer, Berlin-Heidelberg, 1998.

[EL20] L. Ein, R. Lazarsfeld, *The Konno invariant of some algebraic varieties*, European Journal of Mathematics volume 6, 420–429 (2020).

[Enr49] F. Enriques, Federigo, *Le Superficie Algebriche*, Nicola Zanichelli, Bologna, 1949, MR 0031770.

[Fuj87] A. Fujiki, *On the de Rham Cohomology Group of a Compact Kahler Symplectic Manifold*, Adv. Stud. Pure Math. 10 (1987), 105-165.

[Ful79] W. Fulton, J. Hansen, *A connectedness theorem for projective varieties with applications to intersections and singularities of mappings*, Annals of Mathematics. 110 (1): 159–166, 1979.

[Ful84] W. Fulton, *Intersection Theory*, Ergebnisse der Mathematik und ihrer Grenzgebiete 3. Folge, Band 2. Springer (1984).

[GK20] F. Gounelas, A. Kouvidakis, *Geometry of Lines on a Cubic Four-Fold*, International Mathematics Research Notices, 2020.

[GHS03] T. Graber, J. Harris, J. Starr, *Families of rationally connected varieties*, J. Amer. Math. Soc. 16 (2003), p. 57-67.

[GL14] D. Greb, C. Lehn, *Base manifolds for Lagrangian fibrations on hyperkähler manifolds*, Int. Math. Res. Not. IMRN 2014, no. 19, 5483–5487.

[Gri68] P. Griffiths, *Periods of integrals on algebraic manifolds, I, II*, Amer. J. Math. 90, 568-626, 805-865, 1968.

[Gro69] A. Grothendieck, *Hodge's general conjecture is false for trivial reasons*, Inst. Hautes Études Sci. Publ. Math. 29, 95-103, 1969.

[HT84] J. Harris, L. W. Tu, *On symmetric and skew-symmetric determinantal varieties*, Topology, Volume 23, Issue 1, 1984, Pages 71-84.

[Huy14] D. Huybrechts, *Curves and cycles on K3 surfaces (with an appendix by C. Voisin)*, Algebraic Geometry 1 (2014), 69-106.

[Huy99] D. Huybrechts, *Compact hyperkähler manifolds: basic results*, Inventiones mathematicae volume 135, pages 63–113 (1999)

[Huy24] D. Huybrechts, *The geometry of cubic hypersurfaces*, to appear in Cambridge University Press.

[HX22] D. Huybrechts, C. Xu, *Lagrangian fibrations of hyperkaehler fourfolds*, Journal IJM 21 (2022), 921-932

[Hwa08] J. Hwang, *Base manifolds for fibrations of projective irreducible symplectic manifolds*, Invent. Math. 174 (2008), no. 3, 625–644.

[IM08] A. Iliev, L. Manivel, *Cubic hypersurfaces and integrable systems*, Amer. J. Math. 130 (2008), no. 6, 1445–1475.

[Kod64-68] K. Kodaira, (1964), *On the structure of compact complex analytic surfaces. I-IV*, American Journal of Mathematics, 86 (4): 751–798, 88 (3): 682–721, 90 (1): 55–83, 90 (4): 1048–1066. 1964,1966, 1968.

[Kol96] J. Kollár, *Rational curves in algebraic varieties*, Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge (A Series of Modern Surveys in Mathematics), Springer-Verlag Berlin Heidelberg, 1996.

[LSV17] R. Laza, G. Saccà, C. Voisin, *A hyper-Kähler compactification of the Intermediate Jacobian fibration associated with a cubic fourfold*, Acta Mathematica, 218:1 (2017), 55-135.

[Lin16] H.-Y. Lin, *On the Chow group of zero-cycles of a generalized Kummer variety*, Advances in Mathematics 298, 448-472, 2016.

[Man92] L. Manivel, *Théorèmes d'annulation pour les fibrés associés à un fibré ample*, Annali della Scuola Normale Superiore di Pisa, Classe di Scienze 4e série, tome 19, no 4 (1992), p. 515-565.

[Mat99] D. Matsushita, *On fibre space structures of a projective irreducibe symplectic manifold*, Topology, 38 (1999), 79-83.

[Mat15] D. Matsushita, *On base manifolds of lagrangian fibrations*, Sci. China Math, 58(2015), no. 3, 531-542.

[Mat16] D. Matsushita, *On Deformations of Lagrangian Fibrations*, K3 Surfaces and Their Moduli. Progress in Mathematics, vol 315. Birkhäuser, Cham.

[Mar20] O. Martin, *Zero-cycles and measures of irrationality for abelian varieties*, PhD Thesis, 2020.

[Nik80] V.V. Nikulin, *Integral Symmetric Bilinear Forms and Some of their Applications,* Math. USSR Izvestija Vol. 14 (1980), No. 1, 103–167.

[Otw99] A. Otwinowska, *Remarques sur les groupes de Chow des hypersurfaces de petit degré*, C. R. Acad. Sci. Paris Sér. I Math. 329 (1999), no. 1, 51–56.

[Ran92] Z. Ran, *Deformations of manifolds with torsion or negative canonical bundle*, J. Alg. Geom. 1(1992), 279-291.

[Roi80] A. Roitman, *The torsion of the group of 0-cycles modulo rational equivalence*, Ann. Math. 111, 553–569, 1980.

[SV15] M. Shen, Ch. Vial, *The Fourier transforms for certain hyper-Kähler fourfolds*, Memoirs of the American Mathematical Society, Volume: 240; 2015;

[Stap17] D. Stapleton, *The degree of irrationality of very general hypersurfaces in some homogeneous spaces*, Ph.D. thesis, Stony Brook University, 2017.

[vGV16] B. van Geemen, C. Voisin, *On a conjecture of Matsushita*, Int. Math. Res. Notices (2016) Vol. 2016, 3111-3123.

[Voi92] C. Voisin, *Sur la stabilité des sous-variétés lagrangiennes des variétés symplectiques holomorphes*, Complex projective geometry (Trieste, 1989/Bergen, 1989), 294-303, London Math. Soc. Lecture Note Ser., 179, Cambridge Univ. Press, Cambridge, 1992.

[Voi03] C. Voisin. *Théorie de Hodge et géométrie algébrique complexe*, Cours spécialisés, SMF 2003.

[Voi04] C. Voisin, *Intrinsic pseudo-volume forms and K-correspondences*, Proceedings of the Fano Conference, (Eds A.Collino, A. Conte, M. Marchisio), Publication de l'Université de Turin (2004).

[Voi04II] C. Voisin, *Remarks on filtrations on Chow groups and the Bloch conjecture*, Annali di Matematica, 2004.

[Voi08] C. Voisin. On the Chow ring of certain algebraic hyper-Kähler manifolds. Pure Appl. Math. Q. 4 (2008), no. 3, part 2, 613-649.

[Voi10] Claire Voisin, *Hodge loci*, Handbook of moduli, 3:507–546, 2010.

[Voi13] C. Voisin, *The generalized Hodge and Bloch conjectures are equivalent for general complete hypersurfaces*, J. Alg. Geom. 22, 141-174, 2013.

[Voi14] C. Voisin, *Chow rings, decomposition of the diagonal and the topology of families*, Annals of Math. Studies 187, Princeton University Press 2014.

[Voi15] C. Voisin, *Unirational threefolds with no universal codimension* 2 *cycle*, Inventiones mathematicae: Volume 201, Issue 1 (2015), Page 207-237.

[Voi16] C. Voisin, *Remarks and questions on coisotropic subvarieties and 0-cycles of hyper-Kähler varieties*, in "K3 Surfaces and Their Moduli", Proceedings of the Schiermonnikoog conference 2014, C. Faber, G. Farkas, G. van der Geer, Editors, Progress in Math 315, Birkhäuser (2016), 365-399.

[Voi16II] C. Voisin, *The Hodge conjecture. Open problems in mathematics*, 521-543, Springer, (2016).

[Voi17] C. Voisin, *Torsion points of sections of Lagrangian torus fibrations and the Chow ring of hyper-Kähler manifolds*, The Abel Symposium Abelsymposium 2017: Geometry of Moduli, pp 295-326.

[Voi18] C. Voisin, *Chow ring and the gonality of general abelian varieties*, Ann. H. Lebesgue 1 (2018), 313–332.

[Voi21] C. Voisin, *On fibrations and measures of irrationality of hyper-Kähler manifolds*, la Revista de la Unión Matemática Argentina, volume spécial MCA 2021.

[Voi22I] C. Voisin, *Triangle varieties and surface decomposition of hyper-Kähler manifolds*, Recent Developments in Algebraic Geometry: To Miles Reid for his 70th Birthday, edited by Hamid Abban; Gavin Brown; Alexander Kasprzyk; Shigefumi Mori, London Mathematical Society Lecture Note Series Cambridge University Press.

[Voi22II] C. Voisin, *On the Lefschetz standard conjecture for Lagrangian covered hyper-Kähler varieties*, Advances in Mathematics, Volume 396, 2022.

[Yang19] R. Yang, *On irrationality of hypersurfaces in* $\mathbb{P}^{n+1}$, Proc. Amer. Math. Soc. 147 (2019), 971–976.