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Transformations spéciales des quadriques

Thèse présentée et soutenue, le 11 juin 2024 par

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A mi familia.

Résumé

Dans cette thèse, nous étudions les transformations birationnelles spéciales des quadriques lisses. Nous obtenons un résultat de classification en dimensions 3 et 4. Dans ces deux cas, nous démontrons qu'il n'existe qu'un seul exemple. Pour la dimension 3, il est défini par le système linéaire de quadriques passant par une courbe rationnelle normale quartique. Pour la dimension 4, il est défini par le système linéaire de cubiques passant par une surface K3 non minimale de degré 10 avec 2 (-1) -droites disjointes qui n'est contenue dans aucune autre quadrique. Le lieu de base de la transformation inverse est en général une surface lisse du même type. De plus, nous montrons que les surfaces K3 correspondantes sont des partenaires de Fourier-Mukai non isomorphes.

Ces surfaces sont également liées aux cubiques de dimension 4 spéciales. Plus précisément, nous montrons qu'une cubique générale dans le diviseur de Hassett des cubiques spéciales de discriminante 14 contient une telle surface. Il s'agit du premier exemple d'une famille de surfaces non rationnelles caractérisant les cubiques dans ce diviseur.

L'étude des transformations birationnelles spéciales des quadriques est motivée par un exemple décrit par M. Bernardara, E. Fatighenti, L. Manivel, et F. Tanturri, qui ont fourni une liste de 64 nouvelles familles de variétés de Fano de type K3. De nombreux exemples dans leur liste donnent des variétés qui admettent des contractions birationnelles multiples, réalisées comme des éclatements des variétés de Fano le long des surfaces K3 non minimales. La nature des constructions implique que les surfaces K3 ont des catégories dérivées équivalentes. Nous répondons partiellement à la question naturelle : *Pour quelles familles les surfaces K3 correspondantes sont-elles isomorphes, et pour quelles familles ne le sont-elles pas ?*

Mots-clés

Transformations birationnelles spéciales, Variétés de Fano de dimension quatre de type K3, Surfaces K3, Variétés cubiques spéciales de dimension quatre.

Abstract

In this thesis we study special self-birational transformations of smooth quadrics. We obtain a classification result in dimensions 3 and 4. In these two cases, we prove that there is only one example. In the case of dimension 3, it is given by the linear system of quadrics passing through a rational normal quartic curve. In the case of dimension 4, it is given by the linear system of cubic complexes passing through a non-minimal K3 surface of degree 10 with 2 skew (-1) -lines that is not contained in any other quadric. The base locus scheme of the inverse map is in general a smooth surface of the same type. Moreover, we prove that the corresponding pair of K3 surfaces are non-isomorphic Fourier-Mukai partners.

These surfaces are also related to special cubic fourfolds. More precisely, we show that a general cubic in the Hassett divisor of special cubic fourfolds of discriminant 14 contains such a surface. This is the first example of a family of non-rational surfaces characterizing cubics in this divisor.

The study of special birational transformations of quadrics is motivated by an example described by M. Bernardara, E. Fatighenti, L. Manivel, et F. Tanturri, who provided a list of 64 new families of Fano fourfolds of K3 type. Many examples in their list give varieties that admit multiple birational contractions realized as blow-ups of Fano manifolds along non-minimal K3 surfaces. The nature of the constructions implies that the corresponding K3 surfaces have equivalent derived categories. We partially answer the natural question: *for which families the corresponding K3 surfaces are isomorphic, and for which families they are not?*

Keywords

Special birational transformations, Fano fourfolds of K3 type, K3 surfaces, Special cubic fourfolds.

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It took some time to find an adequate topic that I could work on. I believe that this search was also part of the process of realizing what I was doing wrong. Now I can say that I am on the right path, although I have just begun the journey. In this regard, I want to thank you for the flexibility and patience you have shown me throughout these years. I am glad to have been guided by you.

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Table of contents

Résumé	i
Abstract	ii
Acknowledgements	iii
Notation and conventions	1
Introduction (en français)	2
Introduction	8
1 Lattices	14
1.1 Basic lattice theory	14
1.1.1 Examples.	16
1.1.2 Overlattices.	17
1.1.3 Primitive embeddings and unimodular lattices.	18
1.1.4 Existence and uniqueness of even lattices with prescribed signature and discriminant form.	19
1.1.5 Existence and uniqueness of primitive embeddings into fixed even unimodular lattices.	20
1.2 Some topics in the theory of lattices	21
1.2.1 Indefinite binary quadratic forms.	21
1.2.2 Elliptic lattices of rank 2.	25
1.2.3 K3 surfaces, derived equivalences, and lattices.	29
2 Surfaces of type ${}^{\text{II}}Z_E^{10}$	33
2.1 Congruences of lines in \mathbb{P}^3	33
2.2 Sectional genus of a congruence.	34
2.3 Surfaces of degree 10 in Q^4	34
2.4 Special cubic fourfolds of discriminant 14	40
2.5 A note on surfaces of type ${}^{\text{II}}Z_E^{10}$ and quintic Del Pezzo surfaces	43
3 K3 surfaces and Fano fourfolds of K3 type	45
3.1 Introduction	45
3.2 Picard rank 1 cases	47
3.2.1 The family of Fano fourfolds K3-32	47
3.2.2 The family of Fano fourfolds K3-33	51
3.2.3 Geometric action on cycles.	53
3.3 Picard rank 2 cases	55
3.3.1 The family of Fano fourfolds K3-36	55
3.3.2 The family of Fano fourfolds K3-37	56
3.3.3 The family of Fano fourfolds K3-39	56
3.3.4 The family of Fano fourfolds K3-41	56
3.3.5 The family of Fano fourfolds K3-46	57
3.3.6 The family of Fano fourfolds K3-47	57

3.3.7	The family of Fano fourfolds K3-48	57
3.3.8	The family of Fano fourfolds K3-55	58
3.4	Monoidal FM-diagrams	58
4	The cubo-cubic transformation of the smooth quadric fourfold is very special	60
4.1	Basic facts on special rational maps	60
4.2	Numerical constraints	60
4.3	Effective classification	64
4.4	A note on special transformations in dimension 4	72
	Bibliography	77

Notation and conventions

Throughout this thesis we always work over the field \mathbb{C} of complex numbers.

A variety is an integral separated scheme of finite type over \mathbb{C} . Curves, surfaces, threefolds, fourfolds, or hypersurfaces in some k -dimensional variety X are understood to be varieties of dimension 1, 2, 3, 4, and $k - 1$, respectively. We reserve the words conic, quadric, cubic, quartic, etc., for any closed subscheme in some projective space \mathbb{P}^k with the corresponding degree. In particular, these subschemes are not necessarily reduced or irreducible.

We say that a subvariety $V \subseteq \mathbb{P}^k$ is non-degenerate if it is not contained in a proper linear subspace.

If V is a variety, we denote by $V^{(p)}$ the blow-up of V at p points. In general, the blow-up of V along a closed subscheme $W \subseteq V$ (or more precisely, along the ideal sheaf corresponding to W) is denoted by $Bl_W(V)$.

We say that a property P of points in some variety M parametrizing a class of objects is *general* if there exists a Zariski open subset U of M such that P is verified for all points in U . We say that P is *very general* if there exist a countable family of closed subsets Z_i of M such that P is verified for all points in the complement of $\bigcup Z_i$.

If E is a globally generated vector bundle on a manifold Y , we denote by $X = \mathcal{L}(Y, E)$ the zero locus of a general section of E in Y . We shall assume that X is a smooth subvariety of Y of codimension $rk(E)$.

Introduction (en français)

Dans cette thèse de doctorat nous classifions les transformations birationnelles spéciales de la quadrique lisse Q^3 de dimension 3 et de la quadrique lisse Q^4 en dimension 4. Ceci est le premier résultat de classification pour les transformations birationnelles spéciales de variétés autres que les espaces projectifs. Le cas le plus intéressant est celui de dimension 4 : il existe un unique exemple d'une transformation birationnelle spéciale $Q^4 \dashrightarrow Q^4$. Il s'agit d'une transformation cubo-cubique dont le lieu de base est une surface K3 non minimale de genre 7. Nous disons que ces surfaces sont de type ${}^{II}Z_E^{10}$. En fait, le lieu de base de l'inverse est également une surface de type ${}^{II}Z_E^{10}$, et les surfaces K3 correspondantes sont des partenaires Fourier-Mukai non isomorphes.

Cette analyse nous amène à démontrer d'autres résultats intéressants. En particulier, nous explorons la relation entre les surfaces de type ${}^{II}Z_E^{10}$ et les 4-variétés cubiques spéciales et nous cherchons d'autres exemples de surfaces K3 dérivées-équivalentes mais non isomorphes dans la construction de nombreuses familles de 4-variétés Fano de type K3 dans l'article [10].

4-variétés cubiques spéciales

Une hypersurface cubique dans \mathbb{P}^k est le lieu des zéros d'un polynôme homogène de degré 3. De nombreuses branches de la géométrie algébrique (par exemple, la théorie de Hodge, les applications des périodes, les surfaces K3, les variétés symplectiques holomorphes irréductibles, et les catégories dérivées) ont connu un progrès considérable grâce aux efforts déployés pour comprendre leur géométrie mystérieuse et fascinante, en particulier en dimension quatre. Cependant, une question simple telle que *la 4-variété cubique est-elle rationnelle ?* reste encore ouverte. Une première étape cruciale vers une réponse est la définition des 4-variétés cubiques spéciales.

Une 4-variété cubique spéciale est une hypersurface cubique lisse dans \mathbb{P}^5 contenant une surface non homologue à une intersection complète. Les cubiques spéciales forment une famille dénombrable de diviseurs \mathcal{C}_d dans l'espace de modules des 4-variétés cubiques lisses indexés par un entier d appelé discriminant. Cet entier dépend du degré de la surface et de son auto-intersection. Nous appelons \mathcal{C}_d le diviseur de Hassett des 4-variétés cubiques spéciales de discriminant d .

Bien que l'on s'attende à ce que la 4-variété cubique très générale ne soit pas rationnelle, aucun exemple non rationnel n'a été trouvé à ce jour. En fait, la célèbre conjecture de Kuznetsov affirme que les 4-variétés cubiques spéciales avec un discriminant "admissible" sont exactement les 4-variétés cubiques rationnelles. Par exemple, les cubiques paramétrés par les diviseurs de Hassett \mathcal{C}_{14} , \mathcal{C}_{26} , \mathcal{C}_{38} et \mathcal{C}_{42} sont connues pour être rationnelles (voir [11], [47] et [48]). Les premiers discriminants admissibles sont précisément $d = 14, 26, 38$ et 42 .

Il est bien connu que \mathcal{C}_{14} peut être caractérisée soit comme la clôture du lieu des cubiques lisses contenant une surface réglée quartique normale rationnelle, soit comme la clôture du lieu des cubiques lisses contenant une surface de Del Pezzo quintique. La géométrie riche des cubiques dans \mathcal{C}_{14} (par exemple, leur caractère rationnel) est classiquement liée à l'existence de ces surfaces particulières. Par conséquent, il est important d'étudier l'existence d'autres surfaces caractérisant une cubique générale de discriminant 14.

Dans le deuxième chapitre de cette thèse, nous étudions les surfaces K3 non minimales de genre 7 de degré 10 dans \mathbb{P}^5 contenant deux (-1) -droites disjointes et contenues dans une

unique hypersurface quadrique lisse, et leur relation avec les cubiques spéciales de discriminant 14. Selon notre terminologie, basée sur celle utilisée par M. Gross dans son classification des surfaces lisses de degré 10 dans la quadrique lisse de dimension quatre à [24], nous disons que ces surfaces sont de type ${}^{II}Z_E^{10}$. Le résultat est le suivant :

Theorem (Théorème 2.4.3). *Une cubique générale $X \in \mathcal{C}_{14}$ contient une surface de type ${}^{II}Z_E^{10}$.*

Ceci est le premier exemple d'une famille de surfaces non rationnelles caractérisant \mathcal{C}_{14} .

La démonstration de ce résultat suit l'approche de H. Nuer dans [39]. L'idée est de considérer le schéma de Hilbert des drapeaux $[S \subseteq X]$, où S est une surface de type ${}^{II}Z_E^{10}$ et X est une 4-variété cubique lisse. La théorie standard des déformations pour les schémas de Hilbert et des arguments de semi-continuité montrent que si nous pouvons exhiber un drapeau explicite $[S \subseteq X]$ tel que $[S]$ est un point lisse de $Hilb_{\mathbb{P}^5}^{\chi(\mathcal{O}_S(n))}$, la fonction $[S] \mapsto h^0(S, \mathcal{I}_{S/\mathbb{P}^5}(3))$ atteint son minimum en $[S]$, et l'égalité $h^0(S, N_{S/X}) = h^0(S, N_{S/\mathbb{P}^5}) + h^0(S, \mathcal{I}_{S/\mathbb{P}^5}(3)) - 55$ est vérifiée, alors il existe un voisinage ouvert de $[S \subseteq X]$ dans le schéma de Hilbert des drapeaux qui est projeté de manière dominante dans le lieu des 4-variétés cubiques spéciales de discriminant 14. Nous utilisons Macaulay2 afin d'exhiber un exemple d'un tel drapeau.

4-variétés de Fano de type K3

La relation étroite entre les 4-variétés cubiques lisses et les surfaces K3 devient évidente en examinant les structures de Hodge correspondantes.

Par exemple, le diamant de Hodge d'une surface K3 est de la forme

$$\begin{array}{cccc} & & 1 & \\ & & 0 & 0 \\ 1 & & 20 & 1 \\ & & 0 & 0 \\ & & 1 & \end{array}$$

et le diamant de Hodge d'un cubique lisse est de la forme

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & & 0 & 0 & & \\ & & & 0 & 1 & 0 & \\ & & 0 & 0 & 0 & 0 & 0 \\ 0 & & 1 & 21 & 1 & 0 & \\ & & 0 & 0 & 0 & 0 & \\ & & 0 & 1 & 0 & & \\ & & 0 & 0 & & & \\ & & 1 & & & & \end{array}$$

De toute évidence, les nombres de Hodge intermédiaires de chaque variété ont une structure semblable. La définition de variété de Fano de type K3 est basée sur cette idée simple et généralise l'exemple des cubiques lisses. Plus précisément, et en dimension quatre, la

principale condition caractérisant les 4-variétés de Fano de type K3 se réduit à l'égalité $h^{3,1} = 1$ et $h^{2,2} > 20$.

Conjecturalement, les variétés symplectiques holomorphes irréductibles (c'est-à-dire, les variétés kählériennes compactes simplement connexes M telles que $H^0(\Omega_M^2) = \mathbb{C}\omega$ pour une 2-forme holomorphe non dégénérée ω) peuvent être associées aux variétés de Fano de type K3. Comme exemple principal, la variété de Fano des droites d'un cubique lisse est une variété symplectique holomorphe irréductible (IHS) de dimension 4 d'après le travail de A. Beauville et R. Donagi dans [8].

Les variétés IHS en dimension 2 sont les surfaces K3. En dimensions paires, et sauf déformation, un nombre remarquablement restreint d'exemples est connu. Comme la rareté d'exemples est le problème principal dans l'étude des variétés IHS, il est très intéressant d'étudier d'autres variétés de Fano de type K3. Cependant, ce n'est que récemment qu'un programme de recherche systématique a été entrepris pour trouver de nouveaux exemples de familles de variétés de Fano de type K3, en particulier en dimension quatre, en dehors des quelques exemples précédemment connus.

Dans l'article récent "Fano fourfolds of K3 type" [10, 2021] de M. Bernardara, E. Fatighenti, L. Manivel, et F. Tanturri, une liste de 64 nouvelles familles de variétés de Fano de type K3 est fournie. Chaque membre de ces familles est défini comme le lieu des zéros d'une section générale d'un fibré vectoriel homogène complètement réductible défini sur un produit de variétés de drapeaux, et leur géométrie est décrite. Cependant, aucune nouvelle structure K3 n'apparaît dans ces exemples. Plus précisément, les structures K3 proviennent de 4-variétés cubiques ou de Gushel-Mukai, ou de surfaces K3.

Dans le troisième chapitre de cette thèse, nous étudions des familles de variétés de Fano de type K3 issues de [10] dont les membres admettent de multiples contractions birationnelles pouvant être réalisées comme des éclatements de variétés de Fano le long des surfaces K3 non minimales. Par exemple, les 4-variétés de la famille K3-33 peuvent être décrits comme l'éclatement de la quadrique lisse de dimension quatre Q^4 le long d'une surface de type ${}^{II}Z_E^{10}$ de deux manières différentes. En fait, dans de nombreux cas (par exemple, K3-33), il a été démontré que les surfaces K3 ont des catégories dérivées équivalentes.

Nous cherchons à répondre à la question naturelle suivante : *pour quelles familles les surfaces K3 sont-elles isomorphes, et pour lesquelles ne le sont-elles pas ?*

Le résultat principal de ce chapitre est

Theorem (Théorème 3.4.1). *Soit X une 4-variété de Fano de type K3 appartenant à la famille K3-N dans [10], où $N = 32, 33, 36, 37, 39, 41, 46, 47, 48$, ou 55. Si $N = 32$ ou 33 et que le rang de Picard des surfaces K3 est 1, alors les surfaces K3 ne sont pas isomorphes. Si $N \neq 32$ et 33 et que le rang de Picard des surfaces K3 est 2, alors les surfaces K3 sont isomorphes.*

Afin de démontrer que les surfaces K3 apparaissant dans K3-32 et K3-33 ne sont pas isomorphes, nous adaptons un magnifique argument de théorie des réseaux de B. Hassett et K.W. Lai de leur article [28] qui tire parti de la structure géométrique des éclatements pour comparer les groupes discriminants des réseaux de Néron-Severi des surfaces K3. Le reste des familles est étudié en suivant une stratégie précise : dans chaque cas, nous décrivons explicitement le réseau de Néron-Severi N de l'une des surfaces K3 de rang de Picard 2 et nous prouvons que **(1)** le genre $\mathcal{G}(N)$ est trivial et **(2)** le morphisme naturel $O(N) \rightarrow O(dN, q_N)$ est surjectif.

Cela suffit pour conclure qu'aucun partenariat de Fourier-Mukai non trivial de la surface K3 étudiée ne peut exister, impliquant ainsi que les surfaces K3 concernées sont isomorphes.

Transformations birationnelles spéciales

Une transformation de Cremona $\varphi : \mathbb{P}^k \dashrightarrow \mathbb{P}^k$ avec un lieu de base lisse et irréductible $\mathfrak{B} \subseteq \mathbb{P}^k$ est dite spéciale. Ces transformations se distinguent par leur structure simple, car elles peuvent être résolues en effectuant l'éclatement de \mathfrak{B} , ce qui donne un diagramme commutatif

$$\begin{array}{ccc} & Bl_{\mathfrak{B}}(\mathbb{P}^k) & \\ \sigma \swarrow & & \searrow \tau \\ \mathbb{P}^k & \xrightarrow{\varphi} & \mathbb{P}^k \end{array}$$

où σ est l'éclatement de \mathbb{P}^k le long de \mathfrak{B} et τ est un morphisme tel que $\varphi = \sigma^{-1}\tau$.

Le type de φ est défini comme la paire ordonnée d'entiers (d_1, d_2) telle que φ et φ^{-1} sont définies par un sous-système linéaire de $|\mathcal{O}_{\mathbb{P}^k}(d_1)|$ et $|\mathcal{O}_{\mathbb{P}^k}(d_2)|$, respectivement. On dit que φ est une transformation quadro-quadrique si son type est $(2, 2)$, une transformation quadro-quintique si son type est $(2, 5)$, une transformation cubo-cubique si son type est $(3, 3)$, et ainsi de suite.

L'étude des transformations de Cremona spéciales est un problème classique. Si des invariants numériques, tels que la dimension du lieu de base ou le type de la transformation sont fixés, une classification peut être possible, mais souvent difficile.

Les premiers résultats généraux ont été obtenus par B. Crauder et S. Katz dans [14, 1989], qui ont classifié les transformations de Cremona spéciales avec un lieu de base de dimension au plus 2. Telles transformations sont assez rares. Par exemple, il n'existe que deux types de transformations de Cremona de \mathbb{P}^4 avec une surface lisse S comme lieu de base : une transformation cubo-quadrique où S est une surface réglée elliptique quintique, et une transformation quardo-quartique où S est une surface déterminantale de degré 10. Peu de temps après ce succès, l'intérêt pour le problème de classification a commencé à croître et de nouveaux résultats ont été trouvés dans [19, 1989], [15, 1991], et [30, 1992]. Par exemple, les transformations de Cremona spéciales de \mathbb{P}^5 avec une 3-variété lisse comme lieu de base ont été classifiées par L. Ein et N. Shepherd-Barron dans [19], lançant ainsi la quête de la classification complète en dimension 3.

Plus tard, le problème de classification a été généralisé pour considérer des transformations birationnelles spéciales $\mathbb{P}^k \dashrightarrow Z$, où Z est une variété suffisamment bien comportée. A. Alzati et J. C. Sierra ont obtenu un résultat de classification sous l'hypothèse que Z est une variété de Fano première et que le lieu de base a une dimension au plus 2 dans [1, 2016], et G. Staglianò a étudié les transformations birationnelles spéciales quadratiques, c'est-à-dire, de type $(2, n)$, dans des variétés factorielles Z dans [52, 2012] et [53, 2013].

Sur la base de ses travaux antérieurs et de ceux de L. Ein et N. Shepherd-Barron, G. Staglianò a achevé la classification des transformations de Cremona spéciales avec un lieu de base de dimension 3 dans [54, 2016] et [55, 2019]. Récemment seulement, il a obtenu la classification des transformations birationnelles spéciales $\mathbb{P}^k \dashrightarrow Z$ sous l'hypothèse que Z est une variété d'intersection complète factorielle et que le lieu de base a une dimension au plus 3 dans [56, 2020].

Dans tous les cas, seulement les transformations birationnelles spéciales dont le domaine est \mathbb{P}^k ont été considérées. Le problème suivant à considérer est celui de classer les applications birationnelles spéciales dont le domaine est une variété rationnelle autre que l'espace projectif. En fait, certains exemples potentiels de transformations birationnelles spéciales entre variétés rationnelles peuvent être trouvés dans des travaux récents liés au problème de classification des 4-variétés de Fano. Par exemple, les 4-variétés de la famille K3-33 dans [10] induisent des transformations cubo-cubiques birationnelles de Q^4 .

Cet exemple particulier conduit naturellement au problème de la classification des transformations birationnelles spéciales des quadriques lisses Q^k . Dans le quatrième chapitre de cette thèse, nous classifions les transformations birationnelles spéciales de Q^3 et Q^4 . Sans surprise, il n'y a qu'un exemple dans chaque dimension :

Theorem (Théorème 4.2.7). *Soit $\varphi : Q^3 \dashrightarrow Q^3$ une transformation birationnelle spéciale. Alors φ est une transformation quadro-quadrique, et son lieu de base C est une courbe rationnelle normale quartique.*

Réciproquement, soit $C \subseteq Q^3$ une courbe rationnelle normale quartique. Alors, le système linéaire des quadriques dans Q^3 passant par C induit une transformation birationnelle spéciale $\varphi : Q^3 \dashrightarrow Q^3$ avec C comme lieu de base. De plus, le lieu de base de φ^{-1} est également une courbe rationnelle normale quartique.

Nous pouvons penser à cette transformation spéciale comme à l'analogie de la transformation spéciale cubo-cubique de \mathbb{P}^3 , classifiée par S. Katz dans son article "The cubo-cubic transformation of \mathbb{P}^3 is very special" [34, 1987]. Le cas en dimension quatre est beaucoup plus intéressant.

Theorem (Théorème 4.3.14). *Soit $\varphi : Q^4 \dashrightarrow Q^4$ une application birationnelle spéciale. Alors φ est une transformation cubo-cubique, et son lieu de base est une surface S de type ${}^{II}Z_E^{10}$.*

Inversement, soit $S \subseteq Q^4$ une surface de type ${}^{II}Z_E^{10}$ et soit S_0 son modèle minimal. Alors le système linéaire $|H_S + K_S|$ induit un morphisme $S \rightarrow \mathbb{P}^7$ réalisant l'éclatement de S_0 en deux points p et p' . Si H_{S_0} est la classe d'hyperplan de $S_0 \subseteq \mathbb{P}^7$ et qu'il n'existe aucune courbe elliptique lisse $\mathfrak{E} \subseteq S_0$ de degré 5 telle que $\langle H_{S_0}, \mathfrak{E} \rangle \subseteq \text{Pic}(S_0)$ soit saturé et que les points p et p' appartiennent à une fibre (éventuellement singulière) de la fibration elliptique associée de S_0 , alors le système linéaire de cubiques passant par S induit une transformation birationnelle spéciale $\varphi : Q^4 \dashrightarrow Q^4$ avec S comme lieu de base. De plus, si $\text{rk Pic}(S_0) = 1$, alors le lieu de base T de φ^{-1} est également une surface lisse de type ${}^{II}Z_E^{10}$. En fait, les surfaces K3 S_0 et T_0 sont des partenaires non isomorphes de Fourier-Mukai. Plus précisément, T_0 est l'espace de modules de faisceaux stables de S_0 avec un vecteur de Mukai $(2, H_{S_0}, 3)$.

Il y a de nombreux points de comparaison entre cette transformation birationnelle spéciale de Q^4 et une transformation de Cremona de \mathbb{P}^4 étudiée par B. Hassett et K. W. Lai dans [28, 2018]. Dans leur article, ils ont classifié les transformations de Cremona de \mathbb{P}^4 dont le lieu de base est donné par une surface ayant uniquement des points doubles ordinaires comme singularités, et ils ont montré qu'il n'y a qu'un tel exemple. Nous pouvons voir la transformation cubo-cubique spéciale de Q^4 comme la version lisse de leur exemple.

La stratégie de la classification dans le cas de dimension 4 est la suivante. Si nous supposons que $Q^4 \xrightarrow{\varphi} Q^4$ est une transformation birationnelle spéciale, alors il existe une résolution $Q^4 \xleftarrow{\sigma} Bl_{\mathfrak{B}}(Q^4) \xrightarrow{\tau} Q^4$, où σ est l'éclatement de Q^4 le long du lieu de base \mathfrak{B} de φ et τ est un morphisme tel que $\varphi = \sigma^{-1}\tau$. En fait, le morphisme τ est également un éclatement. Par conséquent, nous pouvons décrire le diviseur canonique $K_{Bl_{\mathfrak{B}}(Q^4)}$ de deux manières. Comme $\text{codim}(\mathfrak{B}) \geq 2$, les équations résultantes impliquent que φ est une transformation cubo-cubique et que \mathfrak{B} est une surface, que nous notons S . Les formules de théorie de intersection classiques pour les éclatements lisses donnent naissance à une série de relations arithmétiques entre différents invariants géométriques de S . Les résultats sur les congruences de droites dans \mathbb{P}^3 du deuxième chapitre nous permettent de déduire que la surface S a degré 10, son genre sectionnel est 7, son diviseur canonique a degré 2, sa caractéristique d'Euler topologique est 26, et S ne peut pas être contenue dans un complexe quadratique. Selon la classification de M. Gross dans [24] et la Définition 2.3.3, cela implique que S est une surface de type $II Z_E^{10}$.

Inversement, nous pouvons prouver qu'une telle surface produit une transformation birationnelle spéciale $Q^4 \dashrightarrow Z$, où Z est une hypersurface quadrique dans \mathbb{P}^5 de rang au moins 3. La principale difficulté est de montrer que Z est lisse et que si S est assez général, alors le lieu de base T de la transformation inverse φ^{-1} est également lisse. Pour ce faire, nous étudions la contraction birationnelle $\tau : Bl_S(Q^4) \rightarrow Z$ en utilisant des techniques du Programme de Modèle Minimal (MMP). En particulier, nous montrons que τ est une contraction divisorielle induite par le rayon extrémal R engendré par la classe de la transformée stricte de n'importe quelle trisécante de S sous σ , et le lieu contracté T a dimension 2. Cela implique que la restriction du morphisme $\tau : Exc(R) \rightarrow T$ ne peut avoir que des fibres isolées de dimension 2 au pire. Le célèbre théorème de M. Andreatta et J. A. Wiśniewski classe précisément ce type de fibres dans les contractions birationnelles de Fano-Mori de variétés lisses en dimension quatre (voir Théorème 4.3.8). Selon ce théorème, une fibre spéciale $F = \tau^{-1}(z)$, où $z \in Z$, est soit isomorphe à un plan, soit à une quadrique dans \mathbb{P}^3 de rang au moins 2. Sous les hypothèses appropriées, nous pouvons exclure l'existence de fibres spéciales, ce qui implique que Z et T sont des variétés lisses.

Dans la dernière section, nous renforçons notre résultat de classification en prouvant le résultat suivant :

Theorem (Théorème 4.4.5). *Si une transformation birationnelle $\varphi : Q^4 \dashrightarrow Q^4$ n'est pas un isomorphisme et peut être résolue en éclatant une surface lisse $S \subseteq Q^4$, alors φ est spéciale avec lieu de base S .*

En particulier, les transformations birationnelles de Q^4 induites par les quadriques de la famille K3-33 sont spéciales.

Introduction

In this PhD thesis we classify the special self-birational transformations of the smooth quadric threefold Q^3 and the smooth quadric fourfold Q^4 . This is the first classification result for special self-birational transformations of varieties different to projective spaces. The most interesting case is in dimension four: there is just one example of a special birational transformation $Q^4 \dashrightarrow Q^4$. It is a cubo-cubic map whose base locus scheme is a certain non-minimal K3 surface of degree 10. We say that these surfaces are of type ${}^{II}Z_E^{10}$. In fact, the base locus scheme of the inverse is also a surface of type ${}^{II}Z_E^{10}$, and the corresponding K3 surfaces are non-isomorphic Fourier-Mukai partners.

This analysis brings us to prove other interesting results. In particular, we investigate the relation of surfaces of type ${}^{II}Z_E^{10}$ with special cubic fourfolds and look for other examples of non-isomorphic derived-equivalent K3 surfaces occurring in the construction of many families of Fano fourfolds of K3 type in the paper [10].

Special cubic fourfolds

A cubic hypersurface in \mathbb{P}^k is the zero locus of a homogeneous polynomial of degree 3. Many areas in algebraic geometry (e.g., Hodge theory, period maps, K3 surfaces, IHS manifolds, and derived categories) have experienced tremendous progress thanks to the efforts put in understanding their mysterious and fascinating geometry, especially in dimension four. However, a simple question such as *is the cubic fourfold rational?* still remains open. A crucial first step towards an answer is the definition of special cubic fourfolds.

A special cubic fourfold is a smooth cubic hypersurface in \mathbb{P}^5 containing a surface not homologous to a complete intersection. They form a countable family of divisors \mathcal{C}_d in the moduli space of smooth cubic fourfolds indexed by an integer d called discriminant. This integer depends on the degree of the surface and its self-intersection number. We call \mathcal{C}_d the Hassett divisor of special cubic fourfolds of discriminant d .

Although it is expected that the very general cubic fourfold is non-rational, no example of a provable non-rational cubic has been found as of this day. In fact, the famous Kuznetsov conjecture asserts that special cubic fourfolds with an “admissible” discriminant are exactly the rational cubic fourfolds. For example, cubics parametrized by the Hassett divisors \mathcal{C}_{14} , \mathcal{C}_{26} , \mathcal{C}_{38} , and \mathcal{C}_{42} are known to be rational (see [11], [47], and [48]). The first admissible discriminants d are precisely 14, 26, 38, and 42.

It is well known that \mathcal{C}_{14} can be characterized as either the closure of the locus of smooth cubic fourfolds containing a rational normal quartic scroll or the closure of the locus of smooth cubic fourfolds containing a quintic Del Pezzo surface. The rich geometry of cubics in \mathcal{C}_{14} (e.g., their rationality) is classically tied to the existence of these particular surfaces. Therefore, it is important to investigate the existence of other surfaces characterizing a general cubic of discriminant 14.

In the second chapter of this thesis we study non-minimal K3 surfaces of degree 10 in \mathbb{P}^5 containing two skew (-1) -lines and contained in a smooth quadric hypersurface, but not contained in any other quadric, and their relation to special cubics of discriminant 14. According to our terminology, based on that used by M. Gross in his classification of smooth surfaces of degree 10 in the four-dimensional smooth quadric in [24], we say that these surfaces are of type ${}^{II}Z_E^{10}$. The precise result is the following

Theorem (See Theorem 2.4.3). *A general cubic $X \in \mathcal{C}_{14}$ contains a surface of type ${}^{II}Z_E^{10}$.*

This is the first example of a family on non-rational surfaces characterizing \mathcal{C}_{14} .

The proof of this result follows the approach of H. Nuer in [39]. The idea is to consider the Hilbert scheme of flags $[S \subseteq X]$ where S is a surface of type ${}^{II}Z_E^{10}$ and X is a smooth cubic fourfold. Standard deformation theory for Hilbert schemes and semicontinuity arguments show that if we can exhibit an explicit flag $[S \subseteq X]$ such that $[S]$ is a smooth point of $\text{Hilb}_{\mathbb{P}^5}^{\chi(\mathcal{O}_S(n))}$, the function $[S] \mapsto h^0(S, \mathcal{I}_{S/\mathbb{P}^5}(3))$ achieves its minimum at $[S]$, and the equality $h^0(S, N_{S/X}) = h^0(S, N_{S/\mathbb{P}^5}) + h^0(S, \mathcal{I}_{S/\mathbb{P}^5}(3)) - 55$ is verified, then there exists an open neighborhood of $[S \subseteq X]$ in the Hilbert scheme of flags that is projected dominantly into the locus of special cubic fourfolds of discriminant 14. We use `Macaulay2` in order to exhibit such flag.

Fano fourfolds of K3 type

The close relation between smooth cubic fourfolds and K3 surfaces exists at a very fundamental level. This becomes more transparent by examining the corresponding Hodge structures.

For example, the Hodge diamond of a K3 surface is of the form

$$\begin{array}{cccc} & & & 1 \\ & & 0 & 0 \\ & 1 & 20 & 1 \\ & & 0 & 0 \\ & & & 1 \end{array}$$

and the Hodge diamond of a smooth cubic fourfold is of the form

$$\begin{array}{cccccc} & & & & & 1 \\ & & & & 0 & 0 \\ & & & 0 & 1 & 0 \\ & & 0 & 0 & 0 & 0 \\ & 0 & 1 & 21 & 1 & 0 \\ & & 0 & 0 & 0 & 0 \\ & & 0 & 1 & 0 & 0 \\ & & 0 & 0 & 0 & 0 \\ & & & & & 1 \end{array}$$

Clearly, the middle Hodge numbers of each manifold have a resembling structure. The definition of Fano manifold of K3 type captures this simple idea and generalizes the example of smooth cubic fourfolds. More precisely, and in dimension four, the main condition characterizing Fano fourfolds of K3 type is reduced to the equality $h^{3,1} = 1$.

Conjecturally, irreducible holomorphic symplectic manifolds (i.e., simply connected compact Kähler manifolds M such that $H^0(\Omega_M^2) = \mathbb{C}\omega$ for a non-degenerate holomorphic 2-form ω) can be associated to Fano manifolds of K3 type. As main example, the Fano variety of lines of a smooth cubic fourfold is an irreducible holomorphic symplectic (IHS) manifold of dimension 4 by the work of A. Beauville and R. Donagi in [8].

IHS manifolds in dimension 2 are K3 surfaces. In each fixed even higher dimension, and up to deformation, a remarkably small number of examples are known. Since the lack of examples is the main problem in the study of IHS manifolds, it is very interesting to study similar constructions to the Fano variety of lines for other Fano manifolds of K3 type. However, it is only recently that a conscious research program has been carried out to find new examples of families of Fano manifolds of K3 type, especially in dimension four, other than the few ones previously known.

In the recent paper “Fano fourfolds of K3 type” [10, 2021] by M. Bernardara, E. Fatighenti, L. Manivel, and F. Tanturri, a list of 64 new families of Fano fourfolds of K3 type is provided. Each member from these families is defined as the zero locus of a general section of a completely reducible homogeneous vector bundle defined over a product of flag manifolds and their geometry is described. However, no new K3 structure appears in these examples. More precisely, the K3 structures come from cubic or Gushel-Mukai fourfolds, or K3 surfaces.

In the third chapter of this thesis we study families of Fano fourfolds of K3 type from [10] whose members admit multiple birational contractions that can be realized as blow-ups of Fano manifolds along non-minimal K3 surfaces. For example, fourfolds from the family K3-33 can be described as the blow-up of the four-dimensional smooth quadric Q^4 along a surface of type ${}^{II}Z_E^{10}$ in two different ways. In fact, in many cases (e.g., K3-33) it was shown that the K3 surfaces are derived-equivalent.

We seek to answer the following natural question: *for which families the K3 surfaces are isomorphic, and for which are not?*

The main result of this chapter is

Theorem (See Theorem 3.4.1). *Let X be a Fano fourfold of K3 type belonging to the family K3-N in [10], where $N = 32, 33, 36, 37, 39, 41, 46, 47, 48, \text{ or } 55$. If $N = 32$ or 33 and the Picard rank of the K3 surfaces is 1, then the K3 surfaces are non-isomorphic. If $N \neq 32$ and 33 and the Picard rank of the K3 surfaces is 2, then the K3 surfaces are isomorphic.*

In order to show that the K3 surfaces appearing in K3-32 and K3-33 are non-isomorphic, we adapt a beautiful lattice-theoretic argument of B. Hassett and K.W. Lai from their paper [28] that takes advantage of the geometric structure of the blow-ups to compare the discriminant groups of the Néron-Severi lattices of the K3 surfaces. The rest of the families are studied following a well-delimited strategy: in each case we describe explicitly the Néron-Severi lattice N of one of the K3 surfaces of Picard rank 2 and we prove that **(1)** the genus $\mathcal{G}(N)$ is trivial and **(2)** the natural morphism $O(N) \rightarrow O(dN, q_N)$ is surjective. This is enough to conclude that no non-trivial Fourier-Mukai partners of the K3 surface under study can exist, thus implying that the K3 surfaces involved are isomorphic.

Special birational transformations

A Cremona transformation $\varphi : \mathbb{P}^k \dashrightarrow \mathbb{P}^k$ with a smooth and irreducible base locus scheme $\mathfrak{B} \subseteq \mathbb{P}^k$ is said to be special. These maps stand out due to their simple structure, since they can be “resolved” by performing the blow-up of the base locus scheme \mathfrak{B} , in the sense that we can find a commutative diagram

$$\begin{array}{ccc}
 & Bl_{\mathfrak{B}}(\mathbb{P}^k) & \\
 \sigma \swarrow & & \searrow \tau \\
 \mathbb{P}^k & \xrightarrow{\varphi} & \mathbb{P}^k
 \end{array}$$

where σ is the blow-up of \mathbb{P}^k along \mathfrak{B} and τ is a morphism such that $\varphi = \sigma^{-1}\tau$.

The type of φ is the ordered pair of integers (d_1, d_2) such that φ and φ^{-1} are defined by a linear subsystem of $|\mathcal{O}_{\mathbb{P}^k}(d_1)|$ and $|\mathcal{O}_{\mathbb{P}^k}(d_2)|$, respectively. We convey to say that φ is a quadro-quadric transformation if its type is $(2, 2)$, a quadro-quintic transformation if its type is $(2, 5)$, a cubo-cubic transformation if its type is $(3, 3)$, and so on.

The study of special Cremona transformations is a classical problem. If numerical invariants, such as the dimension of the base locus scheme or the type of the map are fixed, a classification may be possible, although it is most likely challenging.

The first general results were obtained by B. Crauder and S. Katz in [14, 1989], who classified special Cremona transformations with a base locus scheme of dimension at most 2. Transformations with such constraints are quite scarce. For example, there are only two types of Cremona transformations of \mathbb{P}^4 with a smooth surface S as base locus scheme: a cubo-quadric transformation where S is a quintic elliptic scroll and a quarto-quartic transformation where S is a determinantal surface of degree 10. Soon after this success, interest on the classification problem started to grow and new results were found in [19, 1989], [15, 1991], and [30, 1992]. For example, special Cremona transformations of \mathbb{P}^5 with a smooth threefold as base locus scheme were classified by L. Ein and N. Shepherd-Barron in [19], thus setting up the quest for the complete classification in dimension 3.

Later on, the classification problem was generalized to consider special birational transformations $\mathbb{P}^k \dashrightarrow Z$, where Z is a sufficiently nice and well-behaved variety. A. Alzati and J. C. Sierra obtained a classification result under the assumption that Z is a prime Fano manifold and the base locus scheme has dimension at most 2 in [1, 2016] and G. Staglianò studied special quadratic birational transformations, i.e., of type $(2, n)$, into factorial varieties Z in [52, 2012] and [53, 2013].

Based on his previous work and the work of L. Ein and N. Shepherd-Barron, G. Staglianò completed the classification of special Cremona transformations with a base locus scheme of dimension 3 in [54, 2016] and [55, 2019]. Only recently, he obtained the classification of special birational transformations $\mathbb{P}^k \dashrightarrow Z$ under the assumption that Z is a factorial complete intersection variety and the base locus scheme has dimension at most 3 in [56, 2020].

In any case, only special birational transformations with domain \mathbb{P}^k have been considered. The next problem to consider is to classify special birational maps whose domain is a rational variety other than projective space. In fact, some potential examples of special birational transformations between rational varieties can be found in recent works related to the classification problem of Fano fourfolds. For example, fourfolds from the family K3-33 in [10] induce cubo-cubic self-birational transformations of Q^4 .

This particular example leads naturally to the problem of classifying special self-birational transformations of smooth quadrics Q^k . In the fourth chapter of this thesis we classify the special self-birational transformations of Q^3 and Q^4 . Unsurprisingly, there is just one example in each dimension:

Theorem (See Theorem 4.2.7). *Let $\varphi : Q^3 \dashrightarrow Q^3$ be a special birational transformation. Then φ is a quadro-quadric transformation, and its base locus scheme C is a rational normal quartic curve.*

Conversely, let $C \subseteq Q^3$ be any rational normal quartic curve. Then the linear system of quadrics in Q^3 passing through C induces a special birational transformation $\varphi : Q^3 \dashrightarrow Q^3$ with C as base locus scheme. Moreover, the base locus scheme of φ^{-1} is a rational normal quartic curve as well.

We can think of this special transformation as the quadric analogue of the special cubo-cubic transformation of \mathbb{P}^3 , classified by S. Katz in his article “The cubo-cubic transformation of \mathbb{P}^3 is very special” [34, 1987]. The four-dimensional case is much more interesting.

Theorem (See Theorem 4.3.14). *Let $\varphi : Q^4 \dashrightarrow Q^4$ be a special birational map. Then φ is a cubo-cubic transformation, and its base locus scheme is a surface S of type $^{II}Z_E^{10}$.*

Conversely, let $S \subseteq Q^4$ be a surface of type $^{II}Z_E^{10}$ and let S_0 be its minimal model. Then the linear system $|H_S + K_S|$ induces a morphism $S \rightarrow \mathbb{P}^7$ realizing the blow-up of S_0 at two points p and p' . If H_{S_0} is the hyperplane class of $S_0 \subseteq \mathbb{P}^7$ and there is no smooth elliptic curve $\mathfrak{E} \subseteq S_0$ of degree 5 such that $\langle H_{S_0}, \mathfrak{E} \rangle \subseteq \text{Pic}(S_0)$ is saturated and the points p and p' belong to a (possibly singular) fiber of the associated elliptic fibration of S_0 , then the linear system of cubics passing through S induces a special birational transformation $\varphi : Q^4 \dashrightarrow Q^4$ with S as base locus scheme. If moreover $\text{rkPic}(S_0) = 1$, then the base locus scheme T of φ^{-1} is a smooth surface of type $^{II}Z_E^{10}$ as well. In fact, the K3 surfaces S_0 and T_0 are non-isomorphic Fourier-Mukai partners. More precisely, T_0 is the moduli space of stable sheaves of S_0 with Mukai vector $(2, H_{S_0}, 3)$.

There are many points of comparison between this special self-birational transformation of Q^4 and a Cremona transformation of \mathbb{P}^4 studied by B. Hassett and K. W. Lai in [28, 2018]. In their paper they classified Cremona transformations of \mathbb{P}^4 whose base locus scheme is given by a surface with only transverse double points as singularities and they showed that there is just one such example. We can see the special cubo-cubic transformation of Q^4 as the smooth version of their example.

The strategy of the classification in the four-dimensional case is as follows. If we assume that $Q^4 \xrightarrow{\varphi} Q^4$ is a special birational transformation, then there exists a resolution $Q^4 \xleftarrow{\sigma} \text{Bl}_{\mathfrak{B}}(Q^4) \xrightarrow{\tau} Q^4$, where σ is the blow-up of Q^4 along the base locus scheme \mathfrak{B} of φ and τ is a morphism such that $\varphi = \sigma^{-1}\tau$. In fact, the morphism τ is also a blow-up map. Hence, we can compare the canonical divisor $K_{\text{Bl}_{\mathfrak{B}}(Q^4)}$ in two ways. Since $\text{codim}(\mathfrak{B}) \geq 2$, the resulting equations imply that φ is a cubo-cubic transformation and \mathfrak{B} is a surface, which we now denote by S . Now, classical intersection-theoretic formulas for smooth blow-ups give rise to a bunch of arithmetic relations among different geometric invariants of S . The results on congruences of lines in \mathbb{P}^3 from the second chapter allow us to deduce from the arithmetic relations that the surface S has degree 10, its sectional genus is 7, its canonical divisor has degree 2, its topological Euler characteristic is 26, and S cannot be contained in a quadratic complex. According to the classification of M. Gross in [24] and Definition 2.3.3, this implies that S is a surface of type $^{II}Z_E^{10}$.

Conversely, we can prove that any such surface produces a special birational transformation $Q^4 \dashrightarrow Z$, where Z is a quadric hypersurface in \mathbb{P}^5 of rank at least 3. The principal difficulty

is to show that Z is smooth, and that if S is general enough, then the base locus scheme T of the inverse map φ^{-1} is also smooth. To do this we study the birational contraction $\tau : Bl_S(Q^4) \rightarrow Z$ using techniques from the Minimal Model Program (MMP). In particular, we show that τ is a divisorial contraction induced by the extremal ray R of the Mori cone of $Bl_S(Q^4)$ spanned by the class of the strict transform of any trisecant line of S under σ , and the contracted locus T has dimension 2. This implies that the restricted morphism $\tau : Exc(R) \rightarrow T$ can only have isolated fibers of dimension 2 at worst. The celebrated theorem of M. Andreatta and J. A. Wiśniewski classifies precisely these type of fibers in Fano-Mori birational contractions of smooth fourfolds (see Theorem 4.3.8). According to this theorem, a special fiber $F = \tau^{-1}(z)$, where $z \in Z$, is either isomorphic to a plane or to a quadric in \mathbb{P}^3 of rank at least 2. Under the appropriate hypotheses, we can rule out the existence of special fibers, implying smoothness of our construction.

In the last section, we strengthen our classification result by proving the following result:

Theorem (See Theorem 4.4.5). *If a birational transformation $\varphi : Q^4 \dashrightarrow Q^4$ is not an isomorphism and can be resolved by blowing up a smooth surface $S \subseteq Q^4$, then φ is special with base locus scheme S .*

In particular, the self-birational transformations of Q^4 induced by fourfolds from the family K3-33 are special.

1. Lattices

In this chapter we give an introduction to the theory of lattices, focusing on some applications to the study of lattices of rank 2. The main reference used for this chapter is the article [38] by V. Nikulin. For other reference, consult the survey article [17] by I. Dolgachev. A textbook treatment focused on applications to the theory of K3 surfaces can be found in [31, Chapter 14].

1.1 Basic lattice theory

A lattice Λ is a free \mathbb{Z} -module of finite rank together with a non-degenerate symmetric bilinear form $(\cdot, \cdot) : \Lambda \times \Lambda \rightarrow \mathbb{Z}$. We denote $r = \text{rank}(\Lambda)$.

The lattice Λ is called even if the square $x^2 := (x, x)$ of any $x \in \Lambda$ is an even integer. Otherwise we will say that Λ is odd.

The bilinear form (\cdot, \cdot) induces an injection of finite index $\iota : \Lambda \rightarrow \Lambda^* := \text{Hom}_{\mathbb{Z}}(\Lambda, \mathbb{Z})$ and the determinant of the corresponding matrix M with respect to any \mathbb{Z} -basis and its dual is called the discriminant of Λ . This number is denoted by $\det(\Lambda)$. For example, the discriminant of Λ^* is $\det(\Lambda^*) = \det(\Lambda)^{-1}$. We say that Λ is unimodular if ι is an isomorphism, or equivalently, if its discriminant is ± 1 .

Note that M is nothing but the intersection matrix associated to (\cdot, \cdot) with respect to a chosen \mathbb{Z} -basis and $(x, y) = x^t M y$ for all $x, y \in \Lambda$. Evidently, a change of integral basis in Λ via some matrix $S \in GL_r(\mathbb{Z})$ changes the intersection matrix M by $S^t M S$.

The (orthogonal) direct sum of two lattices Λ_1 and Λ_2 is the lattice defined as the direct sum of the underlying groups $\Lambda_1 \oplus \Lambda_2$ together with the symmetric bilinear form defined by $((x_1, x_2), (y_1, y_2)) = (x_1, y_1)_1 + (x_2, y_2)_2$, where $x_1, y_1 \in \Lambda_1$ and $x_2, y_2 \in \Lambda_2$.

A morphism of lattices $\Lambda \rightarrow \Lambda'$ is just a group morphism that respects the respective bilinear forms. A bijective morphism of lattices for which the inverse is also a morphism is called isometry, while an injective morphism is called embedding. If $\Lambda_1 \hookrightarrow \Lambda$ is an embedding of finite index then the formula $\det(\Lambda_1) = \det(\Lambda)[\Lambda : \Lambda_1]^2$ holds.

Let $\Lambda \hookrightarrow \Lambda'$ and $\Lambda \hookrightarrow \Lambda''$ be a pair of lattice embeddings. We consider them as isomorphic if there exists an isometry $\Lambda' \rightarrow \Lambda''$ making these maps conjugate to each other.

An embedding of lattices $\Lambda_1 \hookrightarrow \Lambda$ is said to be primitive if its cokernel is torsion free. For example, for any sublattice $\Lambda_1 \subseteq \Lambda$ the set $\Lambda_1^\perp := \{x \in \Lambda : (x, \Lambda_1) = 0\} \subseteq \Lambda$, called the orthogonal complement of Λ_1 in Λ , is a primitive sublattice of Λ such that $\Lambda_1 \cap \Lambda_1^\perp = 0$. On the other hand, the embedding $\Lambda_1 \oplus \Lambda_1^\perp \hookrightarrow \Lambda$ is of finite index, but it is not necessarily primitive. In general, $\Lambda_1 \subseteq \Lambda_1^{\perp\perp} \subseteq \Lambda$. However, equality holds if and only if Λ_1 is a

primitive sublattice of Λ . In fact, $\Lambda_1^{sat} := \Lambda_1^{\perp\perp}$ is the minimal primitive sublattice of Λ containing Λ_1 . We call it the saturation of Λ_1 . It is also characterized as the sublattice $\{x \in \Lambda : nx \in \Lambda_1 \text{ for some } 0 \neq n \in \mathbb{Z}\}$. Clearly, $rk(\Lambda_1) = rk(\Lambda_1^{sat})$, i.e., the inclusion $\Lambda_1 \subseteq \Lambda_1^{sat}$ is of finite index.

For $K = \mathbb{Q}, \mathbb{R}$, or \mathbb{C} , we extend the bilinear pairing (\cdot, \cdot) to $\Lambda_K := \Lambda \otimes K$. By non-degeneracy, the intersection matrix M can be diagonalized over \mathbb{R} so that it has only 1's and -1 's on the diagonal. We thus define the signature of Λ to be $sign(\Lambda) := (n_+, n_-)$, where n_{\pm} is the number of ± 1 's on the diagonal. We also define the index of Λ as the integer $\tau(\Lambda) := n_+ - n_-$. The lattice Λ is called definite if n_+ or n_- is zero. More precisely, it is positive definite if n_- is zero and negative definite if n_+ is zero. If there is at least a 1 and a -1 on the diagonal, then it is called indefinite.

The dual Λ^* of Λ can be identified with $\{x \in \Lambda_{\mathbb{Q}} : (x, \Lambda) \subseteq \mathbb{Z}\} \subseteq \Lambda_{\mathbb{Q}}$ and the composition $\Lambda \xrightarrow{\iota} \Lambda^* \hookrightarrow \Lambda_{\mathbb{Q}}$ can be identified with the canonical inclusion $\Lambda \hookrightarrow \Lambda_{\mathbb{Q}}$. The cokernel of ι is called the discriminant group of Λ , and we denote it by $d\Lambda$. This is a finite group of order equal to the absolute value of the discriminant of Λ , i.e., $\#d\Lambda = |det(\Lambda)|$. Hence, the group $d\Lambda$ is trivial if and only if Λ is a unimodular lattice. We denote by $l(\Lambda)$ the minimal number of generators of the group $d\Lambda$.

We can also induce a finite symmetric bilinear form $b_{\Lambda} : d\Lambda \times d\Lambda \rightarrow \frac{\mathbb{Q}}{\mathbb{Z}}$ of the discriminant group $d\Lambda$ by the formula $b_{\Lambda}(x + \Lambda, y + \Lambda) = (x, y) \bmod \mathbb{Z}$. If Λ is an even lattice we can even induce a finite quadratic form $q_{\Lambda} : d\Lambda \rightarrow \frac{\mathbb{Q}}{2\mathbb{Z}}$ by the formula $q_{\Lambda}(x + \Lambda) = x^2 \bmod 2\mathbb{Z}$. The maps b_{Λ} and q_{Λ} are known as the discriminant bilinear form of Λ and the discriminant quadratic form of Λ , respectively. These maps are non-degenerate and are related by the following polarization-like equation:

$$\frac{1}{2}(q_{\Lambda}(x + y) - q_{\Lambda}(x) - q_{\Lambda}(y)) = b_{\Lambda}(x, y)$$

Here $\frac{1}{2} : \frac{\mathbb{Q}}{2\mathbb{Z}} \rightarrow \frac{\mathbb{Q}}{\mathbb{Z}}$ is the natural isomorphism and $x, y \in \Lambda$. In particular, we see that b_{Λ} is determined by q_{Λ} .

We say that two lattices Λ_1 and Λ_2 have the same genus if $sign(\Lambda_1) = sign(\Lambda_2)$ and $\Lambda_1 \otimes \mathbb{Z}_p$ is isometric to $\Lambda_2 \otimes \mathbb{Z}_p$ for all primes p , where \mathbb{Z}_p denote the p -adic integers. For two even lattices Λ_1 and Λ_2 this is equivalent to the conditions $sign(\Lambda_1) = sign(\Lambda_2)$ and $(d\Lambda_1, q_1)$ is isomorphic to $(d\Lambda_2, q_2)$ by [38, Corollary 1.9.4]. The set of isomorphism classes of even lattices having the same genus as an even lattice Λ is called the genus of Λ and is denoted by $\mathcal{G}(\Lambda)$. It is known to be a finite set by [13, Chapter 9, § 4].

Clearly, $(d(\Lambda_1 \oplus \Lambda_2), b_{\Lambda_1 \oplus \Lambda_2})$ is naturally isomorphic to $(d\Lambda_1, b_{\Lambda_1}) \oplus (d\Lambda_2, b_{\Lambda_2})$ and $(d(\Lambda_1 \oplus \Lambda_2), q_{\Lambda_1 \oplus \Lambda_2})$ is naturally isomorphic to $(d\Lambda_1, q_{\Lambda_1}) \oplus (d\Lambda_2, q_{\Lambda_2})$ in the even case.

For any isometry $\varphi : \Lambda_1 \rightarrow \Lambda_2$ of lattices (resp. even lattices) we induce an isomorphism $d\Lambda_1 \rightarrow d\Lambda_2$ preserving the corresponding discriminant bilinear forms (resp. discriminant quadratic forms) defined by $f \bmod \Lambda_1 \mapsto (\varphi^*)^{-1}(f) \bmod \Lambda_2$. However, we shall still denote it by φ . Under the identification of Λ^* with the subgroup of $\Lambda_{\mathbb{Q}}$ mentioned earlier, this isomorphism is clearly given by $x \bmod \Lambda_1 \mapsto \varphi_{\mathbb{Q}}(x) \bmod \Lambda_2$. In particular, we can define a natural group morphism $\mathcal{O}(\Lambda) \rightarrow \mathcal{O}(d\Lambda, b_{\Lambda})$ (resp. $\mathcal{O}(\Lambda) \rightarrow \mathcal{O}(d\Lambda, q_{\Lambda})$). Here $\mathcal{O}(\Lambda)$ is the group of isometries of the lattice Λ and $\mathcal{O}(d\Lambda, b_{\Lambda})$ (resp. $\mathcal{O}(d\Lambda, q_{\Lambda})$) is the group of automorphisms of the discriminant group $d\Lambda$ preserving the discriminant bilinear form b_{Λ} (resp. discriminant quadratic form q_{Λ}) of Λ . Note that in the even case we have $\mathcal{O}(d\Lambda, q_{\Lambda}) \subseteq \mathcal{O}(d\Lambda, b_{\Lambda})$.

The following theorem gives a criterion to distinguish when two lattices have the same discriminant forms.

1.1.1 Theorem. *Let Λ_1 and Λ_2 be two lattices (resp. even lattices). Then $(d\Lambda_1, b_1)$ is isomorphic to $(d\Lambda_2, b_2)$ (resp. $(d\Lambda_1, q_1)$ is isomorphic to $d\Lambda_2, q_2$) if and only if $\Lambda_1 \oplus L_1$ is isometric to $\Lambda_2 \oplus L_2$, where L_1 and L_2 are some unimodular lattices (resp. even unimodular lattices).*

Proof. See [38, Theorem 1.3.1]. □

1.1.1 Examples.

i) We denote by \mathbb{Z} the rank 1 lattice whose generator's square is 1. The orthogonal direct sum $\mathbb{Z}^{\oplus n}$ is also denoted by I_n .

ii) The hyperbolic plane U is the lattice $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

Here we are doing an abuse of notation; the underlying group of the lattice U is \mathbb{Z}^2 , and its intersection matrix with respect to the canonical generators $e_1 = (1, 0)$ and $e_2 = (0, 1)$ is the one displayed above. Thus $e_1^2 = e_2^2 = 0$ and $e_1e_2 = e_2e_1 = 1$. A similar convention applies to the upcoming lattices.

The discriminant of U is equal to -1 , so it has signature $(1, 1)$. This lattice is quite special and appears throughout the theory. For example, it has the property that for any embedding $U \hookrightarrow \Lambda$ we have a decomposition $\Lambda = U \oplus U^\perp$ given by $\alpha \mapsto (e_1\alpha)e_2 + (e_2\alpha)e_1 + \alpha'$, where α' is orthogonal to U .

iii) The E_8 lattice is given by

$$\begin{pmatrix} 2 & -1 & & & & & & \\ -1 & 2 & -1 & & & & & \\ & -1 & 2 & -1 & -1 & & & \\ & & -1 & 2 & 0 & & & \\ & & -1 & 0 & 2 & -1 & & \\ & & & & -1 & 2 & -1 & \\ & & & & & -1 & 2 & -1 \\ & & & & & & -1 & 2 \end{pmatrix}$$

This rank 8 lattice is even, unimodular, positive definite, and it has discriminant equal to 1.

iv) For a lattice Λ and an integer m , we define $\Lambda(m)$ to be the lattice with the same underlying group as Λ , but where the bilinear form is scaled by m . In other words, the intersection matrix of $\Lambda(m)$ is the intersection matrix of Λ multiplied by m .

The discriminant of this lattice is $\det(\Lambda(m)) = m^r \det(\Lambda)$. This can be easily seen from the following exact sequence

$$0 \rightarrow \frac{\Lambda}{m\Lambda} \rightarrow d\Lambda(m) \rightarrow d\Lambda \rightarrow 0$$

Notice that this sequence also shows that $d\Lambda(m) = \frac{\Lambda}{m\Lambda}$ whenever Λ is a unimodular lattice.

v) The K3 lattice is by definition $E_8(-1)^{\oplus 2} \oplus U^{\oplus 3}$. This is an even, unimodular lattice of signature $(3, 19)$ and discriminant equal to -1 . For any K3 surface S , the second integral cohomology group $H^2(S, \mathbb{Z})$ becomes a lattice by endowing it with the usual intersection

pairing of topological cycles on S . In fact, $H^2(S, \mathbb{Z})$ is isometric to $E_8(-1)^{\oplus 2} \oplus U^{\oplus 3}$, hence the name. We also have the so-called Néron-Severi lattice $NS(S)$ of S consisting of algebraic cohomology classes in $H^2(S, \mathbb{Z})$, and the transcendental lattice $T(S)$ of S , defined as the orthogonal lattice to $NS(S)$ in $H^2(S, \mathbb{Z})$.

vi) The Mukai lattice. This is defined as $\Lambda \oplus U$, where Λ is the K3 lattice and U is the hyperbolic lattice. This is an even and unimodular lattice of signature $(4, 20)$ and discriminant equal to 1. For any K3 surface S , the group $\tilde{H}(S, \mathbb{Z}) := H^0(S, \mathbb{Z}) \oplus H^2(S, \mathbb{Z}) \oplus H^4(S, \mathbb{Z})$ endowed with the pairing $\alpha\beta := \alpha_2\beta_2 - \alpha_0\beta_4 - \alpha_4\beta_0$ is a lattice abstractly isometric to the Mukai Lattice. Together with $H^2(S, \mathbb{Z})$, $NS(S)$, and $T(S)$, the Mukai lattice plays a fundamental role in the study of K3 surfaces.

1.1.2 Overlattices.

Let Λ be a lattice (resp. even lattice). An overlattice (resp. even overlattice) of Λ is another lattice (resp. even lattice) Λ' such that there is finite index embedding $\Lambda \hookrightarrow \Lambda'$. In some sense, this notion is opposite to the notion of primitive embedding.

Let $H_{\Lambda'} := \frac{\Lambda'}{\Lambda}$. We have a chain of embeddings $\Lambda \hookrightarrow \Lambda' \hookrightarrow \Lambda'^* \hookrightarrow \Lambda^*$. It induces another chain of embedding $H_{\Lambda'} \hookrightarrow \frac{\Lambda'^*}{\Lambda} \hookrightarrow d\Lambda$ and $\frac{\Lambda'^*}{H_{\Lambda'}} = d\Lambda'$. Also note that the discriminant bilinear form b (resp. discriminant quadratic form q) of Λ is identically zero when it is restricted to the subgroup $H_{\Lambda'}$.

More generally, we say that a subgroup H of $d\Lambda$ is isotropic if b (resp. q) is zero when restricted to H . For such a group we define $\Lambda_H \subseteq \Lambda^* \subseteq \Lambda_{\mathbb{Q}}$ to be the inverse image of H under the quotient $\Lambda^* \rightarrow d\Lambda$. In particular, it contains Λ and the restriction of the \mathbb{Q} -valued symmetric bilinear form on $\Lambda_{\mathbb{Q}}$ to Λ_H defines a lattice (resp. even lattice) structure. We thus have the following

1.1.2 Proposition. *The assignment $\Lambda' \mapsto H_{\Lambda'}$ is a bijective correspondence between overlattices (resp. even overlattices) of Λ and isotropic subgroups of $d\Lambda$. Its inverse is given by $H \mapsto \Lambda_H$. Moreover, we have $H_{\Lambda'}^{\perp} = \frac{\Lambda'^*}{\Lambda}$, and thus $\frac{H_{\Lambda'}^{\perp}}{H_{\Lambda'}} = d\Lambda'$, and $b|_{H_{\Lambda'}^{\perp}} \bmod H_{\Lambda'} = b'$ (resp. $q|_{H_{\Lambda'}^{\perp}} \bmod H_{\Lambda'} = q'$), where b' is the discriminant bilinear form (resp. q' is the discriminant quadratic form) of Λ' . \square*

We give now a criterion for the existence of an isometry extending a fixed isometry of Λ in the context of overlattices.

1.1.3 Proposition. *Consider two overlattices (resp. even overlattices) Λ' and Λ'' of Λ and an isometry $\varphi \in O(\Lambda)$. Then φ extends to an isometry between Λ' and Λ''*

$$\begin{array}{ccc} \Lambda' & \xrightarrow{\cong} & \Lambda'' \\ \uparrow & & \uparrow \\ \Lambda & \xrightarrow{\varphi} & \Lambda \end{array}$$

if and only if the induced automorphism φ of $d\Lambda$ restricts to an isomorphism of the isotropy groups $H_{\Lambda'}$ and $H_{\Lambda''}$.

$$\begin{array}{ccc} d\Lambda & \xrightarrow{\varphi} & d\Lambda \\ \uparrow & & \uparrow \\ H_{\Lambda'} & \xrightarrow{\cong} & H_{\Lambda''} \end{array}$$

In particular, two overlattice embeddings (resp. even overlattice embeddings) $\Lambda \hookrightarrow \Lambda'$ and $\Lambda \hookrightarrow \Lambda''$ are isomorphic if and only if the isotropy groups $H_{\Lambda'}$ and $H_{\Lambda''}$ are equal.

Proof. See [38, Proposition 1.4.2]. □

1.1.3 Primitive embeddings and unimodular lattices.

Let $\Lambda_1 \rightarrow \Lambda$ be a primitive embedding of a lattice (resp. even lattice) Λ_1 into a unimodular lattice (resp. even unimodular lattice) Λ . Then Λ is an overlattice (resp. even overlattice) of $\Lambda_1 \oplus \Lambda_1^\perp$ with associated isotropic subgroup given by $H_\Lambda = \frac{\Lambda}{\Lambda_1 \oplus \Lambda_1^\perp} \subseteq d\Lambda_1 \oplus d\Lambda_1^\perp$. The primitivity of the embeddings $\Lambda_1, \Lambda_1^\perp \hookrightarrow \Lambda$ is easily seen to be equivalent to the condition that the projections $p_1 : H_\Lambda \rightarrow d\Lambda_1$ and $p_2 : H_\Lambda \rightarrow d\Lambda_1^\perp$ are embeddings. On the other hand, the unimodularity of Λ is equivalent to the condition that these projections are surjective. Therefore the composition $\gamma = p_2(p_1^{-1}) : d\Lambda_1 \rightarrow d\Lambda_1^\perp$ is an isomorphism. Moreover, the isotropy of H_Λ translates to $b_1^\perp \gamma = -b_1$ (resp. $q_1^\perp \gamma = -q_1$), where b_1^\perp (resp. q_1^\perp) is the discriminant bilinear form (resp. discriminant quadratic form) of Λ_1^\perp .

Conversely, consider two lattices (resp. even lattices) Λ_1 and Λ_2 and an isomorphism $\gamma : d\Lambda_1 \rightarrow d\Lambda_2$ such that $b_2 \gamma = -b_1$ (resp. $q_2 \gamma = -q_1$). Then clearly $H := \{(\alpha, \gamma(\alpha)) : \alpha \in d\Lambda_1\}$ is an isotropic subgroup of $d\Lambda_1 \oplus d\Lambda_2$. The corresponding overlattice (resp. even overlattice) Λ_H contains $\Lambda_1 \oplus \Lambda_2$ as a subgroup of index $|H|$. Using the formula $\det(\Lambda_H)|H|^2 = \det(\Lambda_1)\det(\Lambda_2)$ we see that Λ_H is in fact unimodular. We have thus proved

1.1.4 Proposition. *Let Λ_1 and Λ_1^c be lattices (resp. even lattices). A primitive embedding of Λ_1 into some unimodular lattice (resp. even unimodular lattice) for which the orthogonal lattice Λ_1^\perp is isometric to Λ_1^c is determined by an isomorphism $\gamma : d\Lambda_1 \rightarrow d\Lambda_1^c$ such that $b_1^c \gamma = -b_1$ (resp. $q_1^c \gamma = -q_1$). Moreover, two such isomorphisms γ and γ' give rise to isomorphic primitive embeddings if and only if they are conjugate via an automorphism of $d\Lambda_1^c$ that is induced by an isometry of Λ_1^c . □*

1.1.5 Corollary. *Let $\Lambda_1 \hookrightarrow \Lambda$ and $\Lambda_2 \hookrightarrow \Lambda$ be two primitive embeddings of lattices (resp. even lattices) into a unimodular lattice (resp. even unimodular lattice) Λ , and let $\varphi : \Lambda_1 \rightarrow \Lambda_2$ be an isometry. Then φ can be extended to an isometry of Λ*

$$\begin{array}{ccc} \Lambda & \xrightarrow{\cong} & \Lambda \\ \uparrow & & \uparrow \\ \Lambda_1 & \xrightarrow{\varphi} & \Lambda_2 \end{array}$$

if and only if there exists an isometry $\psi : \Lambda_1^\perp \rightarrow \Lambda_2^\perp$ such that $\psi \gamma_1 = \gamma_2 \varphi$ at the level of discriminant groups:

$$\begin{array}{ccc} d\Lambda_1 & \xrightarrow{\varphi} & d\Lambda_2 \\ \gamma_1 \downarrow & & \downarrow \gamma_2 \\ d\Lambda_1^\perp(-1) & \xrightarrow{\psi} & d\Lambda_2^\perp(-1) \end{array}$$

Here $\gamma_i : d\Lambda_i \rightarrow d\Lambda_i^\perp$ is the isomorphism constructed before. □

From now on, we will focus mainly on even lattices.

1.1.4 Existence and uniqueness of even lattices with prescribed signature and discriminant form.

The following theorem is a classical result of J. Milnor on unimodular lattices.

1.1.6 Theorem. *Let n_+ and n_- be non-negative integers.*

- (a) *There exists an odd unimodular lattice Λ of signature (n_+, n_-) if and only if $n_+ + n_- > 0$. If $n_{\pm} > 0$, then Λ is unique up to isometry.*

More precisely, if Λ is an odd indefinite unimodular lattice with $\text{sign}(\Lambda) = (n_+, n_-)$, then Λ is isometric to $I_{n_+} \oplus (I_{n_-}(-1))$.

- (b) *There exists an even unimodular lattice Λ of signature (n_+, n_-) if and only if $n_+ - n_- = 0 \pmod{8}$. If $n_{\pm} > 0$, then Λ is unique up to isometry.*

More precisely, if Λ is an even indefinite unimodular lattice with $\text{sign}(\Lambda) = (n_+, n_-)$, then Λ is isometric to $E_8^{\oplus \frac{\tau(\Lambda)}{8}} \oplus U^{\oplus n_-}$ if $\tau(\Lambda) > 0$ and Λ is isometric to $E_8(-1)^{\oplus \frac{-\tau(\Lambda)}{8}} \oplus U^{\oplus n_+}$ if $\tau(\Lambda) < 0$.

Proof. See [49, Chapter V]. □

A finite symmetric bilinear form is a pair (A, b) , where A is a finite abelian group and $b : A \times A \rightarrow \frac{\mathbb{Q}}{\mathbb{Z}}$ is a symmetric bilinear form. A finite quadratic form is a pair (A, q) , where A is a finite abelian group and $q : A \rightarrow \frac{\mathbb{Q}}{2\mathbb{Z}}$ is a quadratic form, which is a map that satisfies $q(na) = n^2q(a)$ for all $a \in A$ and $n \in \mathbb{Z}$ and for which there exists a symmetric bilinear form $b : A \times A \rightarrow \frac{\mathbb{Q}}{\mathbb{Z}}$ that satisfies $\frac{1}{2}(q(x+y) - q(x) - q(y)) = b(x, y)$ for all $x, y \in A$. We denote by $l(A)$ the minimal number of generators of the group A .

It is known that for every finite symmetric bilinear form (A, b) (resp. finite quadratic form (A, q)) there exists a lattice (resp. even lattice) Λ such that $(d\Lambda, b_{\Lambda})$ is isomorphic to (A, b) (resp. $(d\Lambda, q_{\Lambda})$ is isomorphic to (A, q)). We can define the index $\tau(A, q)$ of a finite quadratic form (A, q) as the index $\tau(\Lambda)$ of an even lattice Λ such that (A, q) is isomorphic to $(d\Lambda, q_{\Lambda})$. However, this is only well defined modulo 8. Indeed, if another even lattice Λ' has the same discriminant (A, q) , then $\Lambda \oplus L_1$ is isometric to $\Lambda' \oplus L_2$ for some even unimodular lattices L_1 and L_2 . But then, $\tau(\Lambda) = \tau(\Lambda') \pmod{8}$ by Theorem 1.1.6.

Another observation is that $\tau(A, -q) = -\tau(A, q) \pmod{8}$. Indeed, assume $(d\Lambda, q_{\Lambda})$ is isomorphic to (A, q) . Then there exists a primitive embedding of Λ into some even unimodular lattice L where the orthogonal complement of Λ is $\Lambda(-1)$, since we trivially have $(d\Lambda(-1), q_{\Lambda(-1)}) = (d\Lambda, -q)$. Then $\tau(\Lambda) + \tau(\Lambda(-1)) = \tau(L) = 0 \pmod{8}$.

Two necessary conditions which are derived from the existence of an even lattice Λ of signature (n_+, n_-) such that $(d\Lambda, q_{\Lambda})$ is isomorphic to (A, b) are clearly $l(A) \leq n_+ + n_-$ and $\tau(A, q) = n_+ - n_- \pmod{8}$. A sufficient condition for the existence and uniqueness of such lattice Λ is the content of the next theorem.

1.1.7 Theorem. *Let n_+ and n_- be non-negative integers and let (A, q) be a finite quadratic form. Then there exists an even lattice Λ of signature (n_+, n_-) and discriminant quadratic form (A, q) if*

$$l(A) + 1 \leq n_+ + n_- \text{ and } \tau(A, q) = n_+ - n_- \pmod{8}.$$

Moreover, if $0 < n_{\pm}$ and $l(A) + 2 \leq n_+ + n_-$, then Λ is the unique even lattice with such signature and discriminant form, up to isometry.

Proof. See [38, Corollary 1.10.2 and Corollary 1.13.3]. \square

The uniqueness part of the previous theorem can be rephrased as follows.

1.1.8 Theorem. *An even indefinite lattice Λ satisfying $l(\Lambda) + 2 \leq rk(\Lambda)$ is unique in its genus, i.e., $\mathcal{G}(\Lambda) = \{\Lambda\}$.* \square

1.1.5 Existence and uniqueness of primitive embeddings into fixed even unimodular lattices.

The following theorem is a classical result of D. James on the existence and uniqueness of primitive embeddings into even unimodular lattices.

1.1.9 Theorem. *There exists a primitive embedding $\Lambda_1 \hookrightarrow \Lambda$ of an even lattice Λ_1 of signature (m_+, m_-) into an even unimodular lattice Λ of signature (n_+, n_-) if $m_+ + m_- \leq \min\{n_+, n_-\}$. Moreover, if the inequality is strict, then the embedding is unique up to an isometry of Λ .*

Proof. See [33]. \square

We can strengthen the last result with the theory described so far in the following way. We know that a primitive embedding of an even lattice Λ_1 of signature (m_+, m_-) into some even unimodular lattice of signature (n_+, n_-) exists if and only if there exists an even lattice Λ_1^c with signature $(n_+ - m_+, n_- - m_-)$ such that $(d\Lambda_1^c, q_1^c)$ is isomorphic to $(d\Lambda_1, -q_1)$. Now, such a lattice Λ_1^c exists if $m_{\pm} \leq n_{\pm}$, $l(\Lambda_1) + 1 \leq (n_+ - m_+) + (n_- - m_-)$ and $\tau(d\Lambda_1, -q_1) = (n_+ - m_+) - (n_- - m_-) \pmod{8}$. The latter condition holds if and only if $n_+ - n_- = 0 \pmod{8}$, since $\tau(d\Lambda_1, -q_1) = -\tau(d\Lambda_1, q_1) = -(m_+ - m_-) \pmod{8}$. Also, note that if $0 < n_{\pm}$ then the unimodular lattice is unique up to isometry.

On the other hand, even if a primitive embedding $\Lambda_1 \hookrightarrow \Lambda$ exists for a fixed even unimodular lattice Λ of signature (n_+, n_-) , we know that it is only unique up to an isometry of Λ if the natural map $O(\Lambda_1^c) \rightarrow O(d\Lambda_1^c, q_1^c)$ is surjective. We have, however, the following result giving a class of even lattices for which this is true.

1.1.10 Theorem. *Let Λ be an even indefinite lattice with $l(\Lambda) + 2 \leq rk(\Lambda)$. Then $O(\Lambda) \rightarrow O(d\Lambda)$ is surjective.*

Proof. See [38, Theorem 1.14.2]. \square

Thus, if we have $m_{\pm} < n_{\pm}$ and $l(\Lambda_1) + 2 = l(\Lambda_1^c) + 2 \leq rk(\Lambda_1^c) = (n_+ - m_+) + (n_- - m_-) = rk(\Lambda) - rk(\Lambda_1)$, then the primitive embedding $\Lambda_1 \hookrightarrow \Lambda$ considered above is unique up to an isometry of Λ .

Combining all the arithmetic conditions on existence and uniqueness, we arrive at the following theorem.

1.1.11 Theorem. *There exists a primitive embedding $\Lambda_1 \hookrightarrow \Lambda$ of an even lattice Λ_1 of signature (m_+, m_-) into an even unimodular lattice Λ of signature (n_+, n_-) if $m_{\pm} < n_{\pm}$ and*

$$l(\Lambda_1) + 2 \leq rk(\Lambda) - rk(\Lambda_1).$$

Moreover, this embedding is unique up to an isometry of Λ . \square

1.1.12 Corollary. *Let $\Lambda_1 \hookrightarrow \Lambda$ be a primitive embedding of an even lattice Λ_1 into an even unimodular lattice Λ such that Λ_1^\perp contains a sublattice isometric to U . Then this embedding is unique up to an isometry of Λ .*

Proof. We have $\Lambda_1^\perp = U \oplus U^\perp$, where U^\perp is the orthogonal of U inside Λ_1 . In particular, the groups $d\Lambda_1, d\Lambda_1^\perp$, and dU^\perp are isomorphic. Then $l(\Lambda_1) = l(\Lambda_1^\perp) = l(U^\perp) \leq rk(U) = rk(\Lambda_1^\perp) - 2 = rk(\Lambda) - rk(\Lambda_1) - 2$. To conclude we use Theorem 1.1.11. \square

1.2 Some topics in the theory of lattices

1.2.1 Indefinite binary quadratic forms.

Binary quadratic forms have a rich history dating back to ancient mathematicians like Diophantus. Their systematic study began in the 18th century with Euler and Lagrange. But it was in Gauss's monumental work "*Disquisitiones Arithmeticae*" in the 19th century where the foundation for a general theory of binary quadratic forms was established, introducing crucial concepts like composition and reduced forms. We will link this old theory with the modern notion of lattice. A reference for this section is [12].

A binary quadratic form is an expression in two variables, say x and y , of the form $f = f(x, y) = ax^2 + bxy + cy^2$, where $a, b, c \in \mathbb{Z}$. We also denote f by $[a, b, c]$.

We define the discriminant of f as $\Delta = b^2 - 4ac$. Note that $\Delta = 0 \pmod{4}$ or $\Delta = 1 \pmod{4}$ and that $b = \Delta \pmod{2}$.

Two binary forms f and g are said to be properly equivalent, denoted by $f \sim g$, if there exist integers α, β, γ , and δ such that $f(\alpha x + \beta y, \gamma x + \delta y) = g(x, y)$, where $\alpha\delta - \beta\gamma = 1$; if this is the case, it is easy to see that the discriminants of f and g are equal. Also, it is evident that \sim is an equivalence relation in the set of binary quadratic forms.

The sign of the discriminant plays a decisive difference in the theory of binary quadratic forms. If a form f has discriminant $\Delta < 0$, then we say that it is definite; if $\Delta = 0$, then it is called degenerate; and if $\Delta > 0$, then it is indefinite. Note that the integers a and c defining a binary quadratic form $[a, b, c]$ have the same sign in the definite case, while they can be positive and/or negative if it is indefinite.

From now on we will only consider indefinite binary quadratic forms.

An indefinite form $[a, b, c]$ is reduced if the following conditions hold:

- (a) $0 < b < \sqrt{\Delta}$.
- (b) $\sqrt{\Delta} - b < 2|a| < \sqrt{\Delta} + b$.

If $[a, b, c]$ is a reduced, then the inequality $\sqrt{\Delta} - b < 2|c| < \sqrt{\Delta} + b$ holds as well.

The set of reduced forms of a given discriminant $\Delta > 0$ is finite. Indeed, the number b in a reduced form $[a, b, c]$ is bounded by definition, and for each of the possible values that b can take, there is a finite number of ways we can write $-\frac{\Delta - b^2}{4}$ as a product ac . Note that this argument also gives a brute-force algorithm that outputs all such reduced forms.

We say that two reduced forms $[a, b, c]$ and $[c, b', a']$ are adjacent if $b + b' = 0 \pmod{2c}$. It can be proved that there is a unique reduced form adjacent to the right and to the left of any given reduced form. Thus, we can produce a so-called cycle of adjacent forms starting from any reduced form $[a, b, c]$.

Any indefinite form is properly equivalent to a reduced form with the same discriminant. Moreover, the set of reduced forms with a given discriminant can be partitioned into cycles of adjacent forms. The key property of these cycles is that two reduced forms are in the same cycle if and only if they are properly equivalent.

In particular, the set of proper equivalence classes of indefinite binary quadratic forms with a given discriminant is finite and each class contains a unique cycle of adjacent reduced forms. We can think of this cycle (or rather, any of its elements) as a representative of the class.

The link between binary quadratic forms and lattice theory comes from the following trivial observations.

Any rank 2 lattice $N = \begin{pmatrix} a & \frac{b}{2} \\ \frac{b}{2} & c \end{pmatrix}$ induces a binary quadratic form $f_N := [a, b, c]$ with discriminant $\Delta = -4\det(N)$. Evidently, $v^2 = f_N(x, y)$ for any $v = xe_1 + ye_2 \in N$. We even say that an integer m is represented by N if $m = f_N(x, y)$ for some integers x and y . Also, if N is definite, degenerate, or indefinite, then so is f_N , and conversely.

Two lattices $N = \begin{pmatrix} a & \frac{b}{2} \\ \frac{b}{2} & c \end{pmatrix}$ and $N' = \begin{pmatrix} a' & \frac{b'}{2} \\ \frac{b'}{2} & c' \end{pmatrix}$ are orientation-preserving isometric, i.e., $N' = S^t N S$ for some $S \in Sl_2(\mathbb{Z})$ if and only if $f_N \sim f_{N'}$. Moreover, if N and N' are even and have the same genus, then f_N and $f_{N'}$ have the same discriminant.

Hence, if N is an even indefinite lattice of rank 2 and we find all reduced binary quadratic forms with discriminant $-4\det(N) > 0$ and the corresponding cycles of adjacent forms, then we can hope to identify much more easily a set of lattices that represent all the isometry classes of $\mathcal{G}(N)$.

Here is an example.

1.2.1 Proposition. *The set of reduced binary quadratic forms with discriminant 80 is given by the union of the following cycles of adjacent forms:*

$$\{[4, 4, -4], [-4, 4, 4]\}, \{[2, 8, -2], [-2, 8, 2]\}, \text{ and } \{[1, 8, -4], [-4, 8, 1]\}.$$

Proof. The set of reduced forms with discriminant 80 can be computed by looking at triples of integers $[a, b, c]$ such that

$$\begin{aligned} 0 < b < 8.9 \sim \sqrt{80}, \\ 8.9 - b < 2|a|, 2|c| < 8.9 + b, \\ \text{and } -\frac{80 - b^2}{4} = ac. \end{aligned}$$

The only possibilities for b are 2, 4, 6 and 8, since b has the same parity as the discriminant. Then ac can be either -19 if $b = 2$, -16 if $b = 4$, -11 if $b = 6$, or -4 if $b = 8$. In particular, a and c must have opposite signs.

We can rule out the cases $b = 2$ and $b = 6$, since 19 and 11 are primes and the inequalities for $|a|$ and $|c|$ are not satisfied. For $b = 4$ the only solutions are $|a| = |c| = 4$, but for $b = 8$ we have that either $|a| = 1$ and $|c| = 4$, or $|a| = |c| = 2$.

With these values of a , b , and c it is easy to see that the partition of the set of reduced forms of discriminant 80 into cycles of adjacent forms is $\{[4, 4, -4], [-4, 4, 4]\}$, $\{[2, 8, -2], [-2, 8, 2]\}$, and $\{[1, 8, -4], [-4, 8, 1]\}$. \square

1.2.2 Corollary. *Let $\Lambda = \begin{pmatrix} 4 & 2 \\ 2 & -4 \end{pmatrix}$ and $\Lambda' = \begin{pmatrix} 2 & 4 \\ 4 & -2 \end{pmatrix}$. Then Λ and Λ' are the only two even lattices of rank 2 and discriminant -20 , up to isometry. For $N = \Lambda$ or Λ' we have $l(N) + 2 \not\leq rk(N)$. However, $\mathcal{G}(N) = \{N\}$ and the map $O(N) \rightarrow O(dN, q_N)$ is surjective.*

Proof. A lattice $N = \begin{pmatrix} a & b \\ b & c \end{pmatrix}$ with discriminant $det(N) = -20$ must be (orientation-preserving) isometric to either $\Lambda = \begin{pmatrix} 4 & 2 \\ 2 & -4 \end{pmatrix}$, $\Lambda' = \begin{pmatrix} 2 & 4 \\ 4 & -2 \end{pmatrix}$, or $\Lambda'' = \begin{pmatrix} 1 & 4 \\ 4 & -4 \end{pmatrix}$ by Proposition 1.2.1.

Note that the lattices Λ , Λ' , and Λ'' are pairwise non-isometric. Indeed, Λ'' is odd, but Λ and Λ' are even, and Λ does not represent 2, whereas Λ' does. From this, it is evident the first claim.

We claim that $d\Lambda = \frac{\mathbb{Z}}{2\mathbb{Z}} \oplus \frac{\mathbb{Z}}{2\mathbb{Z}} \oplus \frac{\mathbb{Z}}{5\mathbb{Z}}$, where the generators in each factor are $f_1 = \frac{e_1}{2}$, $f_2 = \frac{e_2}{2}$, and $f_3 = \frac{2e_1+e_2}{5}$, respectively. Indeed, all three rational vector have integral intersections with e_1 and e_2 , so they lie inside $\Lambda^* \subseteq \Lambda_{\mathbb{Q}}$. It is also clear that their classes in $d\Lambda$ have orders 2, 2, and 5, respectively. Finally, using the intersection matrix of Λ , it is easy to show that the subgroups of $d\Lambda$ generated by each class intersect each other trivially. Note that $l(\Lambda) + 2 = 4 > 2 = rk(\Lambda)$.

We also have $d\Lambda' = \frac{\mathbb{Z}}{2\mathbb{Z}} \oplus \frac{\mathbb{Z}}{2\mathbb{Z}} \oplus \frac{\mathbb{Z}}{5\mathbb{Z}}$. The generators in each factor are $f'_1 = \frac{e_1}{2}$, $f'_2 = \frac{e_2}{2}$, and $f'_3 = \frac{e_1+2e_2}{5}$, respectively. The proof of this is similar. Also, note that $l(\Lambda') + 2 = 4 > 2 = rk(\Lambda')$.

The subgroups $\frac{\mathbb{Z}}{2\mathbb{Z}} \oplus \frac{\mathbb{Z}}{2\mathbb{Z}} \subseteq d\Lambda$ and $\frac{\mathbb{Z}}{2\mathbb{Z}} \oplus \frac{\mathbb{Z}}{2\mathbb{Z}} \subseteq d\Lambda'$ are precisely the subgroups of elements of order 2. We have $q_{\Lambda}(\frac{\mathbb{Z}}{2\mathbb{Z}} \oplus \frac{\mathbb{Z}}{2\mathbb{Z}}) = \{0, 1, -1\} \subseteq \frac{\mathbb{Q}}{2\mathbb{Z}}$ and $q_{\Lambda'}(\frac{\mathbb{Z}}{2\mathbb{Z}} \oplus \frac{\mathbb{Z}}{2\mathbb{Z}}) = \{0, \frac{1}{2}, -\frac{1}{2}\} \subseteq \frac{\mathbb{Q}}{2\mathbb{Z}}$. Thus, even if the groups $d\Lambda$ and $d\Lambda'$ are isomorphic, $(d\Lambda, q_{\Lambda})$ is not isomorphic to $(d\Lambda', q_{\Lambda'})$.

Let $L \in \mathcal{G}(\Lambda)$. Then L is an even lattice of signature $(1, 1)$ and discriminant form (dL, q_L) isomorphic to $(d\Lambda, q_{\Lambda})$. In particular, $det(L) = -20$. Hence, L is isometric to either Λ or Λ' . However, since $(d\Lambda, q_{\Lambda})$ and $(d\Lambda', q_{\Lambda'})$ are non-isomorphic, we must necessarily have that L is isometric to Λ . In other words, $\mathcal{G}(\Lambda) = \{\Lambda\}$. We can prove that $\mathcal{G}(\Lambda') = \{\Lambda'\}$ in a similar way.

Let us deal with the surjectivity of $O(\Lambda) \rightarrow O(d\Lambda, q_{\Lambda})$ first.

We claim that the ring of integers $\frac{\mathbb{Z}[\eta]}{\eta^2-\eta-1}$ is a model for the lattice Λ . Indeed, $\frac{\mathbb{Z}[\eta]}{\eta^2-\eta-1} = \mathbb{Z} \oplus \mathbb{Z}\eta$ can be given a lattice structure by defining the symmetric bilinear pairing $((a+b\eta), (c+d\eta)) \mapsto 2[(a+b\eta)'(c+d\eta) + (a+b\eta)(c+d\eta)'] \in \mathbb{Z}$. Here, we denote $(a+b\eta)' := (a+b) - b\eta$. Then the map taking $(a, b) \in \Lambda$ to $a+b\eta \in \frac{\mathbb{Z}[\eta]}{\eta^2-\eta-1}$ gives an isometry between Λ and $\frac{\mathbb{Z}[\eta]}{\eta^2-\eta-1}$. We will use interchangeably both viewpoints.

We define the conjugate map by $x = a+b\eta \mapsto x' = (a+b\eta)'$, and the multiplication-by- η^2 map by $x = a+b\eta \mapsto \eta^2x = (a+b) + (a+2b)\eta$. These are two special isometries of Λ . Together with $-Id$, they generate the isometry group $O(\Lambda)$ (see [21]). We use the notation $g_1 := -Id$, $g_2 := x \mapsto x'$, and $g_3 := x \mapsto \eta^2x$.

Any group automorphism φ of $d\Lambda = \frac{\mathbb{Z}}{2\mathbb{Z}} \oplus \frac{\mathbb{Z}}{2\mathbb{Z}} \oplus \frac{\mathbb{Z}}{5\mathbb{Z}}$ satisfies relations of the form

$$\begin{aligned} \varphi(f_1) &= af_1 + bf_2 \\ \varphi(f_2) &= cf_1 + df_2 \\ \varphi(f_3) &= ef_3 \end{aligned}$$

for some integers a, b, c, d , and e such that e is coprime with 5. In particular, the subgroups $\frac{\mathbb{Z}}{2\mathbb{Z}} \oplus \frac{\mathbb{Z}}{2\mathbb{Z}}$ and $\frac{\mathbb{Z}}{5\mathbb{Z}}$ contained in $d\Lambda$ are invariant under φ .

An easy computation shows that $f_1^2 = 1, f_2^2 = -1, f_3^2 = \frac{4}{5}$, and $f_1 f_2 = \frac{1}{2}$. In particular, if φ respects the discriminant quadratic form q_Λ , then the following equations hold in $\frac{\mathbb{Q}}{2\mathbb{Z}}$:

$$\begin{aligned} 1 &= a^2 + ab - b^2 \\ -1 &= c^2 + cd - d^2 \\ \frac{4}{5} &= e^2 \frac{4}{5}. \end{aligned}$$

Since $\gcd(e, 5) = 1$, the last equation implies that e is equal to 1 or 4 mod 5, so φ restricts to $\pm Id$ on $\frac{\mathbb{Z}}{5\mathbb{Z}}$. Also, the solutions mod 2 of the first two equations give the set of six group automorphisms of $\frac{\mathbb{Z}}{2\mathbb{Z}} \oplus \frac{\mathbb{Z}}{2\mathbb{Z}}$. In summary, the isometry group of $d\Lambda$ is $Aut(\frac{\mathbb{Z}}{2\mathbb{Z}} \oplus \frac{\mathbb{Z}}{2\mathbb{Z}}) \oplus \{\pm Id_{\frac{\mathbb{Z}}{5\mathbb{Z}}}\}$.

Hence, in order to show that $O(\Lambda) \rightarrow O(d\Lambda, q_\Lambda)$ is surjective, it will be enough to show that the six automorphisms of $\frac{\mathbb{Z}}{2\mathbb{Z}} \oplus \frac{\mathbb{Z}}{2\mathbb{Z}} \subseteq d\Lambda$ are induced by the restriction of some isometry of Λ . Indeed, $g_1 = -Id \in O(\Lambda)$ acts as $Id = -Id$ on $\frac{\mathbb{Z}}{2\mathbb{Z}} \oplus \frac{\mathbb{Z}}{2\mathbb{Z}}$ and as $-Id$ on $\frac{\mathbb{Z}}{5\mathbb{Z}}$. Therefore, we can compose $g_1 \in O(d\Lambda, q_\Lambda)$ with any isometry of $d\Lambda$ without changing anything in the subgroup $\frac{\mathbb{Z}}{2\mathbb{Z}} \oplus \frac{\mathbb{Z}}{2\mathbb{Z}}$, but changing the sign on the subgroup $\frac{\mathbb{Z}}{5\mathbb{Z}}$ if necessary.

Here are six such automorphisms of $d\Lambda$:

$$\begin{aligned} g_3 &= \begin{pmatrix} 1 & 1 & \\ 1 & 0 & \\ & & -1 \end{pmatrix}, g_3^2 = \begin{pmatrix} 0 & 1 & \\ 1 & 1 & \\ & & 1 \end{pmatrix}, g_3^3 = \begin{pmatrix} 1 & 0 & \\ 0 & 1 & \\ & & -1 \end{pmatrix}, \\ g_2 &= \begin{pmatrix} 1 & 1 & \\ 0 & 1 & \\ & & -1 \end{pmatrix}, g_3 g_2 = \begin{pmatrix} 1 & 0 & \\ 1 & 1 & \\ & & 1 \end{pmatrix}, g_2 g_3 = \begin{pmatrix} 0 & 1 & \\ 1 & 0 & \\ & & 1 \end{pmatrix}. \end{aligned}$$

Now we show the surjectivity of $O(\Lambda') \rightarrow O(d\Lambda', q_{\Lambda'})$.

As before, any group automorphism φ of $d\Lambda' = \frac{\mathbb{Z}}{2\mathbb{Z}} \oplus \frac{\mathbb{Z}}{2\mathbb{Z}} \oplus \frac{\mathbb{Z}}{5\mathbb{Z}}$ satisfies relations of the form

$$\begin{aligned} \varphi(f'_1) &= a f'_1 + b f'_2 \\ \varphi(f'_2) &= c f'_1 + d f'_2 \\ \varphi(f'_3) &= e f'_3 \end{aligned}$$

for some integers a, b, c, d , and e such that e is coprime with 5. In particular, the subgroups $\frac{\mathbb{Z}}{2\mathbb{Z}} \oplus \frac{\mathbb{Z}}{2\mathbb{Z}}$ and $\frac{\mathbb{Z}}{5\mathbb{Z}}$ contained in $d\Lambda'$ are invariant under φ .

We compute $f_1'^2 = \frac{1}{2}, f_2'^2 = -\frac{1}{2}, f_3'^2 = \frac{2}{5}$, and $f_1' f_2' = 1$. In particular, if φ respects the discriminant quadratic form $q_{\Lambda'}$, then the following equations hold in $\frac{\mathbb{Q}}{2\mathbb{Z}}$:

$$\begin{aligned} \frac{1}{2} &= \frac{a^2 - b^2}{2} \\ -\frac{1}{2} &= \frac{c^2 - d^2}{2} \\ \frac{2}{5} &= e^2 \frac{2}{5}. \end{aligned}$$

Since $\gcd(e, 5) = 1$, the last equation implies that e is equal to 1 or 4 mod 5, so φ restricts to $\pm Id$ on $\frac{\mathbb{Z}}{5\mathbb{Z}}$. Also, the solutions mod 2 of the first two equations show that φ restricts to $Id = -Id$ on $\frac{\mathbb{Z}}{2\mathbb{Z}} \oplus \frac{\mathbb{Z}}{2\mathbb{Z}}$. In summary, the isometry group of $d\Lambda'$ is $\{\pm Id\}$. This implies the claimed surjectivity. \square

1.2.2 Elliptic lattices of rank 2.

Let $n, k \in \mathbb{N}$. We denote by $\Lambda_{n,k}$ the rank 2 lattice $\begin{pmatrix} 2k & n \\ n & 0 \end{pmatrix}$. Recall that this means that $e_1 = (1, 0) \in \mathbb{Z}^2$ and $e_2 = (0, 1) \in \mathbb{Z}^2$ satisfy $e_1^2 = 2k$, $e_1 e_2 = e_2 e_1 = n$, and $e_2^2 = 0$. The lattice $\Lambda_{n,k}$ is thus even indefinite of signature $(1, 1)$ and its discriminant is $-n^2$. We call these lattices elliptic. Intuitively, they represent the Néron-Severi lattice of a K3 surface of Picard rank 2 possessing an elliptic fibration over \mathbb{P}^1 : the class of the elliptic fiber is isotropic by the genus formula, and its intersection product with a polarization of degree $2k$ is n .

The following proposition gives a practical criterion to help us recognize if two elliptic lattices are isometric.

1.2.3 Proposition. *Let $n, k_1, k_2 \in \mathbb{Z}$ with $n \neq 0$. Then Λ_{n,k_1} is isometric to Λ_{n,k_2} if $k_1 = k_2 \pmod n$ or $k_1 k_2 = 1 \pmod n$. Moreover, the condition is also necessary if $\gcd(k_1, n) = 1$ or $\gcd(k_2, n) = 1$.*

Proof. If $k_1 = k_2 + ne$ for some $e \in \mathbb{Z}$, then $A = \begin{pmatrix} 1 & 0 \\ -e & 1 \end{pmatrix} \in GL(2, \mathbb{Z})$ verifies $\Lambda_{n,k_2} = A^t \Lambda_{n,k_1} A$; if $k_1 k_2 = 1 + ne$ for some $e \in \mathbb{Z}$, then $B = \begin{pmatrix} k_2 & n \\ -e & -k_1 \end{pmatrix} \in GL(2, \mathbb{Z})$ verifies $\Lambda_{n,k_2} = B^t \Lambda_{n,k_1} B$. For the second part, see the proof [57, Lemma 3.2 (iii)]. \square

The isometry group of $\Lambda_{n,k}$ has at most four elements. The precise structure of this group is described in

1.2.4 Proposition. *Let $n, k \in \mathbb{Z}$ with $n \neq 0$ and define $n' := \frac{n}{m}$ and $k' := \frac{k}{m}$, where $m := \gcd(n, k)$. If $k' = \pm 1 \pmod{n'}$, then there exists an isometry J of $\Lambda_{n,k}$ that swaps the two unique (up to a sign) primitive isotropic vectors F and $F' = n'H - k'F$.*

The isometry group $O(\Lambda_{n,k})$ is given by

$$\begin{aligned} \frac{\mathbb{Z}}{2\mathbb{Z}} \oplus \frac{\mathbb{Z}}{2\mathbb{Z}} &= \{\pm Id, \pm J\} \text{ if } k = 0 \text{ or } k'^2 = 1 \pmod{n'}, \text{ and} \\ \frac{\mathbb{Z}}{2\mathbb{Z}} &= \{\pm Id\} \text{ otherwise.} \end{aligned}$$

Proof. The condition $k' = \pm 1 \pmod{n'}$ implies the condition $k'^2 = 1 \pmod{n'}$. Hence, $k'^2 - n'e = 1$ for some $e \in \mathbb{Z}$. Then J is defined by the transformation $\begin{pmatrix} k' & n' \\ -e & -k' \end{pmatrix} \in GL(2, \mathbb{Z})$. In fact, $J^2 = Id$, i.e., J is an involution. For the second part see [59, Lemma 4.6]. \square

The discriminant group of an elliptic lattice is easy to write down. We keep denoting by e_1 and e_2 the images of these vectors in the discriminant group.

1.2.5 Proposition. *Let $n, k \in \mathbb{Z}$ with $n \neq 0$. The discriminant group $d\Lambda_{n,k}$ is isomorphic to $\frac{\mathbb{Z}}{u\mathbb{Z}} \oplus \frac{\mathbb{Z}}{v\mathbb{Z}}$, where $v = \gcd(2k, n)$ and $u = \frac{n^2}{v}$. The generators of order u and v are $f = \frac{\frac{n}{v}e_1 - \frac{2k}{v}e_2}{u}$ and $g = \frac{e_2}{v}$, respectively. Furthermore, if Λ is a lattice with the same genus as $\Lambda_{n,k}$, then Λ is isometric to $\Lambda_{n,l}$, where $\gcd(2l, n) = \gcd(2k, n)$.*

Proof. See [37, Lemma 3.3]. \square

For cyclic discriminant groups, we have

1.2.6 Proposition. *Let $n, k, k_1, k_2 \in \mathbb{Z}$ with $n \neq 0$.*

- (a) *$\gcd(2k, n) = 1$ if and only if the discriminant group $d\Lambda_{n,k}$ is cyclic.*
 (b) *If $\gcd(2k_1, n) = \gcd(2k_2, n) = 1$, then Λ_{n,k_1} has the same genus as Λ_{n,k_2} if and only if there exists $a \in \mathbb{Z}$ such that $k_1 = a^2 k_2 \pmod{n^2}$ and $\gcd(a, n) = 1$.*

Proof. See [57, Lemma 3.2]. □

We now give a comprehensive study of lattices of the form $\Lambda_{n,k}$ for small values of n .

1.2.7 Lemma. *Let p a prime number and $n < m$. The number of group automorphisms of $\mathbb{Z}_{p^m} \oplus \mathbb{Z}_{p^n}$ is given by $\tau(p^m)\tau(p^n)p^{2n}$, where τ is the Euler totient function.*

Proof. A group automorphism $\varphi : \mathbb{Z}_{p^m} \oplus \mathbb{Z}_{p^n} \rightarrow \mathbb{Z}_{p^m} \oplus \mathbb{Z}_{p^n}$ can be written as

$$\begin{aligned}\varphi(f) &= af + bg \\ \varphi(g) &= p^{m-n}cf + dg\end{aligned}$$

where $f \in \mathbb{Z}_{p^m}$ and $g \in \mathbb{Z}_{p^n}$ are primitive generators, and a, b, c , and d are integers such that a and d are coprime to p . This representation is unique up to demanding $a \in \mathbb{Z}_{p^m}$ and $b, c, d \in \mathbb{Z}_{p^n}$, in which case we have $\tau(p^m)$ choices for a , $\tau(p^n)$ for d and p^n choices for both b and c . □

1.2.8 Remark. In matrix notation a group automorphism φ as in the proof of Lemma 1.2.7 is represented by $\begin{pmatrix} a & p^{m-n}c \\ b & d \end{pmatrix}$. Here we take coefficients modulo p^m in the first row, and coefficients modulo p^n in the second row.

1.2.9 Theorem. *The following is a classification of elliptic lattices $\Lambda_{n,k}$, their discriminant groups and isometry groups thereof, where $n = 1, 2, 3, 4$, and 5.*

- *There is only one isomorphism class of lattices $\Lambda_{1,k}$. A representant is the hyperbolic plane $U = \Lambda_{1,0}$. Its discriminant group is trivial, its genus is trivial, its isometry group is $\frac{\mathbb{Z}}{2\mathbb{Z}} \oplus \frac{\mathbb{Z}}{2\mathbb{Z}} = \{\pm Id, \pm J\}$, and the map $O(\Lambda_{1,0}) \rightarrow O(d\Lambda_{1,0}, q_{1,0})$ is surjective.*
- *There are two isomorphism classes of lattices $\Lambda_{2,k}$. Their representants are $\Lambda_{2,0}$ and $\Lambda_{2,1}$. In both cases, the discriminant group is $\frac{\mathbb{Z}}{2\mathbb{Z}} \oplus \frac{\mathbb{Z}}{2\mathbb{Z}}$, the genus is trivial, the isometry group is $\frac{\mathbb{Z}}{2\mathbb{Z}} \oplus \frac{\mathbb{Z}}{2\mathbb{Z}} = \{\pm Id, \pm J\}$, and the map $O(\Lambda_{2,k}) \rightarrow O(d\Lambda_{2,k}, q_{2,k})$ is surjective.*
- *There are three isomorphism classes of lattices $\Lambda_{3,k}$. Their representants are $\Lambda_{3,0}$, $\Lambda_{3,1}$ and $\Lambda_{3,2}$. The discriminant groups are $\frac{\mathbb{Z}}{3\mathbb{Z}} \oplus \frac{\mathbb{Z}}{3\mathbb{Z}}$ if $k = 0$ and $\frac{\mathbb{Z}}{9\mathbb{Z}}$ otherwise. In all cases the genus is trivial, the isometry group is $\frac{\mathbb{Z}}{2\mathbb{Z}} \oplus \frac{\mathbb{Z}}{2\mathbb{Z}} = \{\pm Id, \pm J\}$, and the map $O(\Lambda_{3,k}) \rightarrow O(d\Lambda_{3,k}, q_{3,k})$ is surjective.*
- *There are four isomorphism classes of lattices $\Lambda_{4,k}$. Their representants are $\Lambda_{4,0}$, $\Lambda_{4,1}$, $\Lambda_{4,2}$, and $\Lambda_{4,3}$. The discriminant groups are $\frac{\mathbb{Z}}{4\mathbb{Z}} \oplus \frac{\mathbb{Z}}{4\mathbb{Z}}$ if $k = 0$ or 2 and $\frac{\mathbb{Z}}{8\mathbb{Z}} \oplus \frac{\mathbb{Z}}{2\mathbb{Z}}$ if $k = 1$ or 3. In all cases the genus is trivial, the isometry group is $\frac{\mathbb{Z}}{2\mathbb{Z}} \oplus \frac{\mathbb{Z}}{2\mathbb{Z}} = \{\pm Id, \pm J\}$, and the map $O(\Lambda_{4,k}) \rightarrow O(d\Lambda_{4,k}, q_{4,k})$ is surjective.*
- *There are four isomorphism classes of lattices $\Lambda_{5,k}$. Their representants are $\Lambda_{5,0}$, $\Lambda_{5,1}$, $\Lambda_{5,2} = \Lambda_{5,3}$, and $\Lambda_{5,4}$. The discriminant groups are $\frac{\mathbb{Z}}{5\mathbb{Z}} \oplus \frac{\mathbb{Z}}{5\mathbb{Z}}$ if $k = 0$ and $\frac{\mathbb{Z}}{25\mathbb{Z}}$ otherwise. The lattices $\Lambda_{5,1}$ and $\Lambda_{5,4}$ are in the same genus, whereas the other lattices have trivial genus. The isometry group of $\Lambda_{5,0}$, $\Lambda_{5,1}$ and $\Lambda_{5,4}$ is $\{\pm Id, \pm J\}$ and the*

isometry group of $\Lambda_{5,2} = \Lambda_{5,3}$ is $\{\pm Id\}$. Moreover, $O(d\Lambda_{5,0}, q_{5,0}) = im(O(\Lambda_{5,0})) \sqcup \left\{ \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}, \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}, \begin{pmatrix} 0 & 2 \\ 3 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 3 \\ 2 & 0 \end{pmatrix} \right\}$ and the map $O(\Lambda_{5,k}) \rightarrow O(d\Lambda_{5,k}, q_{5,k})$ is surjective if $k \neq 0$.

Proof. The case $n = 1$ is trivial. The cases $n = 3$ and $n = 5$ follow from Proposition 1.2.3, Proposition 1.2.4, Proposition 1.2.5, and Proposition 1.2.6. We only need to establish the surjectivity of $O(\Lambda_{n,0}) \rightarrow O(d\Lambda_{n,k}, q_{n,0})$ in both cases. Let φ be an automorphism of $d\Lambda_{n,0} = \frac{\mathbb{Z}}{n\mathbb{Z}} \oplus \frac{\mathbb{Z}}{n\mathbb{Z}}$. Then $\varphi(f) = af + bg$ and $\varphi(g) = cf + dg$, where a, b, c, d are integers, $f = \frac{e_1}{n}$, and $g = \frac{e_2}{n}$. Since φ respects the discriminant bilinear form, and $O(d\Lambda_{n,k}, q_{n,k}) \subseteq O(d\Lambda_{n,k}, b_{n,k})$, we arrive at the following system of congruences:

$$\begin{aligned} ab &= 0 \pmod{n} \\ cd &= 0 \pmod{n} \\ ad + bc - 1 &= 0 \pmod{n} \end{aligned} \tag{1.1}$$

Since n is prime, either $a = 0$ or $b = 0$ and either $c = 0$ or $d = 0$ by the first two congruences in (1.1). Since φ is an automorphism, we get a matrix of the form $\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}$ or $\begin{pmatrix} 0 & c \\ b & 0 \end{pmatrix}$ where $a, d \neq 0$ and $b, c \neq 0$. The last congruence in (1.1) gives $O(d\Lambda_{3,0}, q_{3,0}) = \{\pm Id, \pm J\}$ and $O(d\Lambda_{5,0}, q_{5,0}) = \{\pm Id, \pm J, \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}, \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}, \begin{pmatrix} 0 & 2 \\ 3 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 3 \\ 2 & 0 \end{pmatrix}\}$.

The case $n = 2$ also follows from the same propositions, together with [38, Theorem 3.6.2 and Theorem 3.6.3].

Let us now focus on the case $n = 4$. Note that for any k , the lattice $\Lambda_{4,k}$ is isometric to either $\Lambda_{4,0}$, $\Lambda_{4,1}$, $\Lambda_{4,2}$, or $\Lambda_{4,3}$ by Proposition 1.2.3.

We have the following description of the discriminant groups $d\Lambda_{4,k}$ by Proposition 1.2.5: $d\Lambda_{4,0} = \mathbb{Z}_4 \oplus \mathbb{Z}_4$ has generators $f = \frac{e_1}{4}$ and $g = \frac{e_2}{4}$, $d\Lambda_{4,1} = \mathbb{Z}_8 \oplus \mathbb{Z}_2$ has generators $f = \frac{2e_1 - e_2}{8}$ and $g = \frac{e_2}{2}$, $d\Lambda_{4,2} = \mathbb{Z}_4 \oplus \mathbb{Z}_4$ has generators $f = \frac{e_1}{4}$ and $g = \frac{e_2}{4}$, and $d\Lambda_{4,3} = \mathbb{Z}_8 \oplus \mathbb{Z}_2$ has generators $f = \frac{2e_1 - 3e_2}{8}$ and $g = \frac{e_2}{2}$. These generators determine the corresponding discriminant quadratic forms.

The lattices $\Lambda_{4,0}$ and $\Lambda_{4,2}$ are non-isometric to the lattices $\Lambda_{4,1}$ and $\Lambda_{4,3}$, since their discriminant groups are different. We claim that the discriminant forms of $\Lambda_{4,0}$ and $\Lambda_{4,2}$ are different. Indeed, the subgroup of order 2 in $d\Lambda_{4,0} = \mathbb{Z}_4 \oplus \mathbb{Z}_4 = d\Lambda_{4,2}$ is $N = \{0, 2f, 2g, 2f + 2g\}$. It is easy to see that $q_{4,0}(N) = \{0\} \subseteq \frac{\mathbb{Q}}{2\mathbb{Z}}$ and $q_{4,2}(N) = \{0, 1\} \subseteq \frac{\mathbb{Q}}{2\mathbb{Z}}$. Hence, the lattices $\Lambda_{4,0}$ and $\Lambda_{4,2}$ do not have the same genus. A similar analysis can be done for the lattices $\Lambda_{4,1}$ and $\Lambda_{4,3}$. In this case, the subgroup of elements of order 2 in $d\Lambda_{4,1} = \mathbb{Z}_8 \oplus \mathbb{Z}_2 = d\Lambda_{4,3}$ is $M = \{0, g, 4f, 4f + g\}$. We have $q_{4,1}(M) = \{0, \frac{1}{2}\} \subseteq \frac{\mathbb{Q}}{2\mathbb{Z}}$ and $q_{4,3}(M) = \{0, -\frac{1}{2}\} \subseteq \frac{\mathbb{Q}}{2\mathbb{Z}}$. Hence, the lattices $\Lambda_{4,1}$ and $\Lambda_{4,3}$ do not have the same genus either. This proves that $\Lambda_{4,0}$, $\Lambda_{4,1}$, $\Lambda_{4,2}$, and $\Lambda_{4,3}$ are pairwise non-isometric, and their genus is trivial, i.e., $\mathcal{G}(\Lambda_{4,k}) = \{\Lambda_{4,k}\}$ for all k .

The isometry group of $\Lambda_{4,k}$ is $\{\pm Id, \pm J\}$ by Proposition 1.2.4. Let us show the surjectivity of the natural map $O(\Lambda_{4,k}) \rightarrow O(d\Lambda_{4,k}, q_{4,k})$.

Let $k = 0$ or 2 and let φ be a group automorphism of $d\Lambda_{4,k} = \mathbb{Z}_4 \oplus \mathbb{Z}_4$. Then $\varphi(f) = af + bg$ and $\varphi(g) = cf + dg$ for some integers a, b, c , and d . The generators $f, g \in d\Lambda_{4,k}$ satisfy $f^2 = q_{4,k}(f) = \frac{k}{8}$, $g^2 = q_{4,k}(g) = 0$, and $fg = b_{4,k}(f, g) = \frac{1}{4}$ in $\frac{\mathbb{Q}}{2}$. Since $\varphi \in O(d\Lambda_{4,k}, q_{4,k}) \subseteq$

$O(d\Lambda_{4,k}, b_{4,k})$, we have the following system of equations in $\frac{\mathbb{Q}}{2\mathbb{Z}}$:

$$\begin{aligned}\frac{k}{8} &= f^2 = \varphi(f)^2 = \frac{ka^2 + 4ab}{8} \\ 0 &= g^2 = \varphi(g)^2 = \frac{kc^2 + 4cd}{8}\end{aligned}$$

and the following equation in $\frac{\mathbb{Q}}{\mathbb{Z}}$:

$$\frac{2}{8} = \frac{1}{4} = fg = \varphi(f)\varphi(g) = \frac{kac + 2(ad + bc)}{8}.$$

These equations can be rewritten as a system of congruences:

$$\begin{aligned}k(a^2 - 1) + 4ab &= 0 \pmod{16} \\ kc^2 + 4cd &= 0 \pmod{16} \\ kac + 2(ad + bc - 1) &= 0 \pmod{8}\end{aligned}$$

If $k = 0$ we get the system

$$\begin{aligned}ab &= 0 \pmod{4} \\ cd &= 0 \pmod{4} \\ ad + bc - 1 &= 0 \pmod{4}\end{aligned}\tag{1.2}$$

If $a = 0$, then $b \neq 0$ by the third congruence in (1.2). Hence, $b = 1, 2$, or 3 . But then the solutions are $(b, c) = (1, 1)$ or $(3, 3)$. Consequently, $d = 0$ by the second congruence in (1.2). Now assume that $a \neq 0$. Then b must be 0 or $(a, b) = (2, 2)$ by the first congruence in (1.2). However, the solution $(a, b) = (2, 2)$ does not satisfy the third congruence. Hence, $b = 0$. But then $(a, d) = (1, 1)$ or $(3, 3)$ by the same congruence. Thus, $c = 0$ by the second congruence. Hence, the isometry group $O(d\Lambda_{4,0}, q_{4,0})$ consists of the following elements:

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \text{ and } \begin{pmatrix} 0 & 3 \\ 3 & 0 \end{pmatrix}.$$

These maps are induced by the isometries $Id, -Id, J = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, -J \in O(\Lambda_{4,0})$, respectively (see the proof of Proposition 1.2.4). In particular, the map $O(\Lambda_{4,0}) \rightarrow O(d\Lambda_{4,0})$ is surjective.

If $k = 2$ we get the system

$$\begin{aligned}a^2 + 2ab - 1 &= 0 \pmod{8} \\ c^2 + 2cd &= 0 \pmod{8} \\ ac + ad + bc - 1 &= 0 \pmod{4}\end{aligned}\tag{1.3}$$

Note that a cannot be even by the first congruence in (1.3). On the other hand, c must be even by the second congruence in (1.3). In particular, for odd a the first congruence transforms to $2b = 0 \pmod{8}$ or $6b = 0 \pmod{8}$. Hence, $b = 0 \pmod{8}$ or $b = 4 \pmod{8}$. This forces $b = 0 \pmod{4}$. Then the third congruence in (1.3) becomes $ac + ad - 1 = 0 \pmod{4}$. If $c = 0$, then clearly $(a, d) = (1, 1)$ or $(3, 3)$, and if $c = 2$, then $(a, d) = (1, 3)$ or $(3, 1)$. Hence, the isometry group $O(d\Lambda_{4,2}, q_{4,2})$ consists of the following elements:

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix}, \text{ and } \begin{pmatrix} 3 & 2 \\ 0 & 1 \end{pmatrix}.$$

These maps are induced by the isometries $Id, -Id, J = \begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix}, -J \in O(\Lambda_{4,2})$, respectively (see the proof of Proposition 1.2.4). In particular, the map $O(\Lambda_{4,2}) \rightarrow O(d\Lambda_{4,2})$ is surjective.

Let $k = 1$ or 3 and let φ be a group automorphism of $d\Lambda_{4,k} = \mathbb{Z}_8 \oplus \mathbb{Z}_2$. By Lemma 1.2.7 we have $\varphi(f) = af + bg$ and $\varphi(g) = 4cf + dg$, where a can be $1, 3, 5$ or 7 , d must be 1 , and b and c are either 0 or 1 . The generators $f, g \in d\Lambda_{4,k}$ satisfy $f^2 = q_{4,k}(f) = \frac{-k}{8}, g^2 = q_{4,k}(g) = 0$, and $fg = b_{4,k}(f, g) = \frac{1}{2}$. Since $\varphi \in O(d\Lambda_{4,k}, q_{4,k}) \subseteq O(d\Lambda_{4,k}, b_{4,k})$, we have the following system of equations in $\frac{\mathbb{Q}}{2\mathbb{Z}}$:

$$\begin{aligned} \frac{-k}{8} &= f^2 = \varphi(f)^2 = a^2 \frac{-k}{8} + ab \\ 0 &= g^2 = \varphi(g)^2 = -2kc^2 \end{aligned}$$

and the following equation in $\frac{\mathbb{Q}}{\mathbb{Z}}$:

$$\frac{1}{2} = fg = \varphi(f)\varphi(g) = \frac{a(d - ck)}{2}$$

The second equation above is trivially true, so we will not consider it further. The remaining equations can be rewritten as a system of congruences:

$$\begin{aligned} a^2k - k - 8ab &= 0 \pmod{16} \\ a(d - ck) - 1 &= 0 \pmod{2} \end{aligned} \tag{1.4}$$

From the second congruence in (1.4) we see that if c is odd, then k must be even because d is also odd. This is not the case. Hence, c is even. Now we solve the first congruence in (1.4). For any even b , the solutions are $a = 1, 7, 9$ and 15 . For any odd b , the solutions are $a = 3, 5, 11$, and 13 . Hence, the isometry group $O(d\Lambda_{4,2}, q_{4,2})$ consists of the following elements:

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 7 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 3 & 0 \\ 1 & 1 \end{pmatrix}, \text{ and } \begin{pmatrix} 5 & 0 \\ 1 & 1 \end{pmatrix}.$$

These maps are induced by the isometries $Id, -Id, J = \begin{pmatrix} 1 & 4 \\ 0 & -1 \end{pmatrix}, -J \in O(\Lambda_{4,1})$ and by the isometries $Id, -Id, -J, J = \begin{pmatrix} 3 & 4 \\ -2 & -3 \end{pmatrix} \in O(\Lambda_{4,3})$, respectively (see the proof of Proposition 1.2.4). Hence, the map $O(\Lambda_{4,k}) \rightarrow O(d\Lambda_{4,k}, q_{4,k})$ is surjective for $k = 1$ and 3 . \square

1.2.3 K3 surfaces, derived equivalences, and lattices.

In this section we recall some definitions and results on derived equivalences of K3 surfaces in the view of lattice theory.

Two K3 surfaces S and S' are said to be Fourier-Mukai partners if they are derived equivalent, i.e., there exists an equivalence between the derived categories $D^b(S)$ and $D^b(S')$. The collection of isomorphism classes of Fourier-Mukai partners of S is denoted by $FM(S)$.

Derived equivalence of K3 surfaces can be characterized lattice-theoretically. Indeed, let $T(S)$ be the transcendental lattice of S , i.e., the orthogonal lattice of $NS(S) \subseteq H^2(S, \mathbb{Z})$, and let $H(S, \mathbb{Z})$ be the Mukai lattice of S , endowed with its Hodge structure (see [31, p. 370] for the definition). We have the following well-known result:

1.2.10 Proposition. $D^b(S)$ is equivalent to $D^b(S')$ if and only if $\tilde{H}(S, \mathbb{Z})$ is Hodge-isometric to $\tilde{H}(S', \mathbb{Z})$ if and only if $T(S)$ is Hodge isometric to $T(S')$.

Proof. See [31, Proposition 3.5] and [31, Corollary 3.7]. We reprove the second equivalence to illustrate the application of a result on the unicity of a given lattice embedding.

Clearly, any Hodge isometry between $\tilde{H}(S, \mathbb{Z})$ and $\tilde{H}(S', \mathbb{Z})$ can be restricted to a Hodge isometry between $T(S)$ and $T(S')$. For the converse, consider an arbitrary Hodge isometry $T(S) \xrightarrow{\cong} T(S')$. Observe that we have a hyperbolic plane embedding $U \cong H^0(S, \mathbb{Z}) \oplus H^4(S, \mathbb{Z}) \subseteq T(S)^\perp \subseteq \tilde{H}(S, \mathbb{Z}) \cong \tilde{H}(S', \mathbb{Z})$. In particular, the two embeddings $T(S) \hookrightarrow \tilde{H}(S, \mathbb{Z}) \cong \tilde{H}(S', \mathbb{Z})$ and $T(S) \xrightarrow{\cong} T(S') \subseteq \tilde{H}(S', \mathbb{Z})$ are isomorphic up to an isometry of $\tilde{H}(S', \mathbb{Z})$ by Corollary 1.1.12. This implies that $\tilde{H}(S, \mathbb{Z})$ is Hodge-isometric to $\tilde{H}(S', \mathbb{Z})$. \square

It is known that $FM(S)$ is a finite set (see for example [31, Proposition 3.10]). There is even a lattice-theoretic formula computing the cardinality of this set.

Consider the subgroup of Hodge isometries of the transcendental lattice $T(S)$ of S . We denote it by $O_{Hodge}(T(S)) \subseteq O(T(S))$. Then there is a right action of $O_{Hodge}(T(S))$ on $O(dNS(S))$. It is given by composition of maps via $O_{Hodge}(T(S)) \rightarrow O(dT(S), -q_T) \cong O(dNS(S), q_{NS})$. Also, for each lattice N with the same genus as $NS(S)$, there is a left action of $O(N)$ on $O(dNS(S))$. It is given by composition of maps via $O(N) \rightarrow O(dN, q_N) \cong O(dNS(S), q_{NS})$. We thus have for each $N \in \mathcal{G}(NS(S))$ a double orbit space $O(N) \backslash O(dNS(S)) / O_{Hodge}(T(S))$.

1.2.11 Theorem. *The number of Fourier-Mukai partners of S is given by*

$$|FM(S)| = \sum_{N \in \mathcal{G}(NS(S))} |O(N) \backslash O(dNS(S)) / O_{Hodge}(T(S))|. \quad (1.5)$$

Proof. See [29, Theorem 2.3]. \square

1.2.12 Corollary. *If the genus of $NS(S)$ is trivial, i.e., $\mathcal{G}(NS(S)) = \{NS(S)\}$ and the natural map $O(NS(S)) \rightarrow O(dNS(S))$ is surjective, then $FM(S) = \{S\}$.*

Both conditions are satisfied if the inequality $l(NS(S)) + 2 \leq rk(NS(S))$ holds.

Proof. If $\mathcal{G}(NS(S)) = \{NS(S)\}$, then there is a single term in the formula counting Fourier-Mukai partners (1.5), corresponding precisely to $N = NS(S)$. If, in addition, the map $O(NS(S)) \rightarrow O(dNS(S))$ is surjective, then the action of $O(NS(S)) \times O_{Hodge}(T(S))$ on $O(dNS(S))$ is transitive, implying $\sharp(FM(S)) = 1$. Finally, if $l(NS(S)) + 2 \leq rk(NS(S))$, then both conditions hold by Theorem 1.1.8 and Theorem 1.1.10. \square

1.2.13 Corollary. *We have the following list of other well-known cases where K3 surfaces do not admit non-trivial Fourier-Mukai partners:*

- (a) *If $\rho(S) \geq 12$, then $FM(S) = \{S\}$.*
- (b) *If $\rho(S) \geq 3$ and $\det(NS(S))$ is square-free, then $FM(S) = \{S\}$.*
- (c) *If $S \rightarrow \mathbb{P}^1$ is a Jacobian K3 surface, i.e., an elliptic K3 surface with a section, then $FM(S) = \{S\}$.*

For lower Picard ranks we have the following useful formulas:

(d) Assume that $NS(S) = \mathbb{Z}H$ with $H^2 = 2k$. Then $|FM(S)| = 2\tau(k) - 1$. Here τ is the Euler totient function.

(e) If S is a K3 surface of Picard rank 2, p is a prime $\equiv 1 \pmod{4}$, and $\det(NS(S)) = -p$, then $|FM(S)| = \frac{h(p)+1}{2}$, where $h(p)$ is the class number of $\mathbb{Q}(\sqrt{p})$ ¹.

Proof. Item (a) follows from the computation $l(NS(S)) = l(T(S)) \leq rk(T(S)) = 22 - rk(NS(S)) \leq 22 - 12 = 10 \leq rk(NS(S)) - 2$ and Corollary 1.2.12.

To prove item (b) first note that $l(dNS(S)) = 0$ or 1. Indeed, since $dNS(S)$ is abstractly isomorphic to $\bigoplus \frac{\mathbb{Z}}{m_i \mathbb{Z}}$ we have that $|\det(NS(S))| = \prod m_i$ is square-free. In particular, the m_i are pairwise coprime. Hence, $dNS(S) = \frac{\mathbb{Z}}{(\prod m_i) \mathbb{Z}}$. We have $l(T(S)) + 2 = l(NS(S)) + 2 \leq 3 \leq \rho(S) = rk(N(S)) = rk(H^2(S, \mathbb{Z})) - rk(T(S))$. This implies that the embedding $T(S) \subseteq H^2(S, \mathbb{Z})$ is unique up to Hodge-isometries of $H^2(S, \mathbb{Z})$ by Theorem 1.1.11, and thus the claim follows by Proposition 1.2.10.

To prove item (c) first note that U is embedded in $NS(S) \subseteq T(S)^\perp$, since any fiber F has intersection 1 with the section s of $S \rightarrow \mathbb{P}^1$ and $F^2 = s^2 = 0$. This implies that the embedding $T(S) \subseteq H^2(S, \mathbb{Z})$ is unique up to Hodge-isometries of $H^2(S, \mathbb{Z})$ by Corollary 1.1.12, and thus the claim follows by Proposition 1.2.10.

For item (d) see [40, Proposition 1.10] and for item (e) see [29, Theorem 3.3]. \square

Examples.

1.2.14 Proposition. *Let S be a smooth linear determinantal quartic surface in \mathbb{P}^3 of Picard rank 2. Then S has no non-trivial Fourier-Mukai partners.*

Proof. A smooth linear determinantal quartic surface in \mathbb{P}^3 of Picard rank 2 has $NS(S) = \begin{pmatrix} 4 & 2 \\ 2 & -4 \end{pmatrix}$ by [21, Proposition 2.2]. Then $\mathcal{G}(NS(S)) = \{NS(S)\}$ and the map $O(NS(S)) \rightarrow O(dNS(S), q_{NS})$ is surjective by Corollary 1.2.2. In particular, $FM(S) = \{S\}$ by Corollary 1.2.12. \square

1.2.15 Remark. A general smooth linear determinantal quartic surface in \mathbb{P}^3 has Picard rank 2 by [21].

1.2.16 Proposition. *Let S be a K3 surface of degree 8 in \mathbb{P}^5 of Picard rank 2 and containing a line. Then S has no non-trivial Fourier-Mukai partners.*

Proof. Let $l \subseteq S$ be the line and let H be the polarization of degree 8 of S . The inclusion $L := \langle H, l \rangle \subseteq NS(S)$ has finite index and L has intersection matrix $\begin{pmatrix} 8 & 1 \\ 1 & -2 \end{pmatrix}$. Hence, the discriminant of L is $-17 = \det(L) = \det(NS(S))[NS(S) : L]^2$. Since 17 is prime, this implies that $[NS(S) : L] = 1$, i.e., $NS(S) = L$. Also, we have $17 \equiv 1 \pmod{4}$ and $h(\mathbb{Q}(\sqrt{17})) = 1$. Therefore, $|FM(S)| = \frac{h(\mathbb{Q}(\sqrt{17}))+1}{2} = 1$ by Proposition 1.2.13. Thus, $FM(S) = \{S\}$. \square

1.2.17 Proposition. *Let S be a K3 surface of degree 8 in \mathbb{P}^5 of Picard rank 2 and containing a conic. Then S has no non-trivial Fourier-Mukai partners.*

¹In number theory, the ideal class group of an algebraic number field K is the quotient group F_K/P_K , where F_K is the group of fractional ideals of the ring of integers of K , and P_K is its subgroup of principal ideals. This is a finite group. Its cardinality is called the class number of K .

Proof. Let $c \subseteq S$ be the conic and let H be the polarization of degree 8 of S . The inclusion $L := \langle H, c \rangle \subseteq NS(S)$ has finite index and L has intersection matrix $\begin{pmatrix} 8 & 2 \\ 2 & -2 \end{pmatrix}$. Hence, the discriminant of L is $-20 = \det(L) = \det(NS(S))[NS(S) : L]^2$. In particular, observe that $[NS(S) : L]$ can only be 1 or 2. In the latter case, we have $\det(NS(S)) = -5$. Since 5 is prime, $5 = 1 \pmod{4}$, and $h(\mathbb{Q}(\sqrt{5})) = 1$, we have $\sharp FM(S) = \frac{h(\mathbb{Q}(\sqrt{5}))+1}{2} = 1$ by Proposition 1.2.13, and thus $FM(S) = \{S\}$. Assume now $[NS(S) : L] = 1$. Then $NS(S) = L = \begin{pmatrix} 8 & 2 \\ 2 & -2 \end{pmatrix}$ can be easily shown to be isometric to the lattice $\begin{pmatrix} 2 & 4 \\ 4 & -2 \end{pmatrix}$. In particular, $\mathcal{G}(NS(S)) = \{NS(S)\}$ and the map $O(NS(S)) \rightarrow O(dNS(S), q_{NS})$ is surjective by Corollary 1.2.2. Thus, $FM(S) = \{S\}$ by Corollary 1.2.12. \square

Let $S \rightarrow \mathbb{P}^1$ be an elliptic $K3$ surface. Its multisection index is defined as the minimal positive integer that can be represented as the intersection product of a (hence any) fiber F from the fibration and an arbitrary divisor, i.e., we define it as $\min\{n \in \mathbb{N}^+ : n = DF, D \in NS(S)\}$. For example, the multisection index is 1 if and only if $S \rightarrow \mathbb{P}^1$ is Jacobian, i.e., it has a section.

The Néron-Severi lattice of an elliptic $K3$ surface of Picard rank 2 has a nice description.

1.2.18 Proposition. *Let $S \rightarrow \mathbb{P}^1$ be an elliptic $K3$ surface of Picard rank 2 and multisection index n . Then there exists a polarization H of S such that $HF = n$, where F is the corresponding elliptic fiber. Moreover, the pair (H, F) gives a basis of $NS(S)$. In particular, the Néron-Severi lattice of S is of the form $\Lambda_{n,k}$, where k is some integer.*

Proof. See [50, Lemma 3.3]. \square

Derived-equivalence is a strong condition for elliptic $K3$ surfaces, as the following proposition shows.

1.2.19 Proposition. *Let $S \rightarrow \mathbb{P}^1$ be an elliptic $K3$ surface of Picard rank 2 and multisection index n . Then its Fourier-Mukai partners are also elliptic of Picard rank 2 and multisection index n .*

Proof. Let S' be a Fourier-Mukai partner of S . Then $NS(S)$ and $NS(S')$ have the same genus. In particular, $NS(S')$ represents zero, i.e., S' is elliptic. Let n' be the multisection index of $S' \rightarrow \mathbb{P}^1$. We have $NS(S) = \Lambda_{n,k}$ and $NS(S') = \Lambda_{n',k'}$ for some $d, k' \in \mathbb{Z}$ by Proposition 1.2.18. Also, $U \oplus NS(S)$ is isometric to $U \oplus NS(S')$ by hypothesis. Thus, $-n^2 = \det(NS(S)) = \det(NS(S')) = -n'^2$, i.e., $n = n'$. \square

1.2.20 Proposition. *An elliptic $K3$ surface S of Picard rank 2 and multisection index $n = 2, 3$, or 4 has no non-trivial Fourier-Mukai Partners.*

Proof. The Néron-Severi lattice of S is of the form $\Lambda_{n,k}$ for some integer k by Proposition 1.2.18. In particular, $\mathcal{G}(\Lambda_{n,k}) = \{\Lambda_{n,k}\}$ and the map $O(\Lambda_{n,k}) \rightarrow O(d\Lambda_{n,k}, q_n)$ is surjective by Theorem 1.2.9. Thus, $FM(S) = \{S\}$ by Corollary 1.2.12. \square

2. Surfaces of type $II Z_E^{10}$

This chapter introduces the most important object of study of this PhD thesis, a non-minimal K3 surface of degree 10 contained in $Q^4 \subseteq \mathbb{P}^5$.

2.1 Congruences of lines in \mathbb{P}^3

Here we review some key results from the theory of congruences of lines in \mathbb{P}^3 , which classically refers to the systematic study of smooth surfaces in $Q^4 \cong G(2, 4) = \mathbb{G}(1, \mathbb{P}^3)$. For a good introduction to the subject, see [5].

Let $S \subseteq \mathbb{P}^5$ be a smooth surface of degree \mathfrak{d} contained in Q^4 . We also say that S is a congruence of lines of degree \mathfrak{d} . In the Chow ring of Q^4 we can write $S = \mathfrak{a}P_1 + \mathfrak{b}P_2 \in CH^2(Q^4) = \mathbb{Z}P_1 \oplus \mathbb{Z}P_2$, where P_1 and P_2 are the classes of planes from each of the rulings of Q^4 and \mathfrak{a} and \mathfrak{b} are integers. The pair $(\mathfrak{a}, \mathfrak{b})$ is called the bidegree of S . Under the isomorphism $Q^4 \cong \mathbb{G}(1, \mathbb{P}^3)$, the integer \mathfrak{a} corresponds to the number of lines of the congruence passing through a fixed general point in \mathbb{P}^3 , while the integer \mathfrak{b} corresponds to the number of lines of the congruence contained in a fixed general plane in \mathbb{P}^3 . Clearly, $\mathfrak{d} = \mathfrak{a} + \mathfrak{b}$ and $\mathfrak{a}, \mathfrak{b} \geq 0$.

We denote by H the hyperplane class of $Q^4 \subseteq \mathbb{P}^5$, by H_S its restriction to S , by K_S the canonical divisor of S , and by π the sectional genus of S .

A complex is an effective Cartier divisor in Q^4 . If $0 \neq s \in H^0(Q^4, \mathcal{O}(k))$ and $k \geq 1$, then the zero locus $V(s) \subseteq Q^4$ of the section s is a complex whose degree is defined to be k . Clearly, all complexes are obtained in this way. If $k = 1, 2$ or 3 , we say that $V(s)$ is a linear, a quadratic, or a cubic complex, respectively.

We collect a few formulas computing important invariants of the congruence S .

2.1.1 Proposition. *The total Chern class of N_{S/Q^4} is given by*

$$c(N_{S/Q^4}) = 1 + (K_S + 4H_S)t + (7\mathfrak{d} + 4H_S K_S - c_2(S) + K_S^2)t^2.$$

Proof. See the proof of [5, Proposition 2.1]. □

2.1.2 Corollary. *The Euler characteristic of N_{S/Q^4} is given by*

$$\chi(N_{S/Q^4}) = 6\mathfrak{d} - \mathfrak{a}^2 - \mathfrak{b}^2 + 2(2\pi - 2) + 2\chi(\mathcal{O}_S).$$

Proof. Let \mathbb{V} be a vector bundle over S . Then a Grothendieck-Riemann-Roch computation gives the following formula

$$\chi(\mathbb{V}) = rk(\mathbb{V})\chi(\mathcal{O}_S) + \frac{c_1(\mathbb{V})^2 - c_1(\mathbb{V})K_S}{2} - c_2(\mathbb{V}).$$

The claim follows by taking $\mathbb{V} = N_{S/Q^4}$ and using Proposition 2.1.1. \square

2.1.3 Corollary. *The self-intersection number of K_S is given by*

$$K_S^2 = c_2(S) - 4H_S K_S - 7\mathfrak{d} + \mathfrak{d}^2 - 2\mathfrak{a}\mathfrak{b}.$$

Proof. The second Chern class of N_{S/Q^4} is identified with the intersection product S^2 . Since $S = \mathfrak{a}P_1 + \mathfrak{b}P_2$ in $CH^2(Q^4)$, we have $c_2(N_{S/Q^4}) = S^2 = (\mathfrak{a}P_1 + \mathfrak{b}P_2)^2 = \mathfrak{a}^2 + \mathfrak{b}^2 = \mathfrak{d}^2 - 2\mathfrak{a}\mathfrak{b}$. The claim follows by comparing this equality with the equality $c_2(N_{S/Q^4}) = 7\mathfrak{d} + 4H_S K_S - c_2(S) + K_S^2$ according to Proposition 2.1.1. \square

2.2 Sectional genus of a congruence.

Define the maximal sectional genus $\pi_1(\mathfrak{d})$ of the surface S as the integer

$$\pi_1(\mathfrak{d}) := \begin{cases} \frac{\mathfrak{d}^2 - 4\mathfrak{d} + 8}{8} & \text{if } \mathfrak{d} \equiv 0 \pmod{4}; \\ \frac{\mathfrak{d}^2 - 4\mathfrak{d} + 3}{8} & \text{if } \mathfrak{d} \equiv 1, 3 \pmod{4}; \\ \frac{\mathfrak{d}^2 - 4\mathfrak{d} + 4}{8} & \text{if } \mathfrak{d} \equiv 2 \pmod{4}. \end{cases}$$

The next theorem explains the choice of this name.

2.2.1 Theorem. *If S is non-degenerate and $\mathfrak{d} \geq 9$, then $\pi \leq \pi_1(\mathfrak{d})$.*

Proof. See [23, Theorem 3.6]. \square

The following result gives a lower bound for the arithmetic genus p_a of S in terms of its degree \mathfrak{d} , as well as the existence of certain complexes containing S when the maximal sectional genus is attained.

2.2.2 Theorem. *If S is non-degenerate, $\mathfrak{d} \geq 9$, and $\pi = \pi_1(\mathfrak{d})$, then*

$$p_a \geq \begin{cases} \frac{\mathfrak{d}^3 - 12\mathfrak{d}^2 + 56\mathfrak{d} - 96}{96} & \text{if } \mathfrak{d} \equiv 0 \pmod{4}; \\ \frac{\mathfrak{d}^3 - 12\mathfrak{d}^2 + 41\mathfrak{d} - 30}{96} & \text{if } \mathfrak{d} \equiv 1 \pmod{4}; \\ \frac{\mathfrak{d}^3 - 12\mathfrak{d}^2 + 44\mathfrak{d} - 48}{96} & \text{if } \mathfrak{d} \equiv 2 \pmod{4}; \\ \frac{\mathfrak{d}^3 - 12\mathfrak{d}^2 + 41\mathfrak{d} - 42}{96} & \text{if } \mathfrak{d} \equiv 3 \pmod{4} \end{cases}$$

and S is contained in a quadratic complex. Furthermore, equality holds if and only if S is also contained in an irreducible complex of degree $\lceil \frac{\mathfrak{d}}{4} \rceil$.

Proof. See [24, Theorem 2.6]. \square

2.3 Surfaces of degree 10 in Q^4

The classification of non-degenerate smooth surfaces of degree 10 in the smooth quadric fourfold Q^4 below is due to M. Gross in [24]. The notation $\mathbb{P}^{2(n)}$ stands for the blow-up of \mathbb{P}^2 at n points.

2.3.1 Theorem. *Let $S \subseteq Q^4$ be a non-degenerate smooth surface of degree 10. Then either the bidegree of S is $(3, 7)$, and it is a minimal Enriques surface, or it is of bidegree $(4, 6)$, and it is one of the following types:*

- $\mathbb{P}^{2(12)}$, embedded via $|7L - 2E_1 - \dots - 2E_9 - E_{10} - E_{11} - E_{12}|$. (Z_A^{10})
- A K3 surface blown up in one point. (Z_B^{10})

Or it is of bidegree (5, 5), and it is one of the following types:

- A ruled surface over an elliptic curve with fibers embedded with degree 2, with $\pi = 4, 5$, or 6. (C_A^{10} , C_B^{10} , and C_C^{10})
- $\mathbb{P}^{2(13)}$, embedded via $|7L - 3E_1 - 2E_2 - \dots - 2E_7 - E_8 - \dots - E_{11} - E_{13}|$. (Z_C^{10})
- $\mathbb{P}^{2(17)}$, embedded via $|6L - 2E_1 - 2E_2 - 2E_3 - E_4 - \dots - E_{17}|$. (Z_D^{10})
- A K3 surface blown up in two points. (Z_E^{10})
- A minimal elliptic surface with $p_a = p_g = 2$ and $\pi = 8$. (Z_F^{10})

Proof. See [24, Theorem 1]. □

The following table is extracted from [24, Proposition 4.3, Table 1]. It contains some invariants of non-degenerate surfaces of degree $d \leq 10$ in Q^4 , such as the sectional genus π , the geometric genus p_g , the arithmetic genus p_a , the irregularity q , the self-intersection number of the canonical divisor K , etc.

Surface	d	π	p_g	p_a	q	K^2	KH	c_2	bidegree
Veronese	4	0	0	0	0	9	-6	3	(1, 3)
F_0, F_2	4	0	0	0	0	8	-6	4	(2, 2)
Del Pezzo	5	1	0	0	0	5	-5	7	(2, 3)
Elliptic scroll	6	1	0	-1	1	0	-6	0	(3, 3)
Del Pezzo	6	1	0	0	0	6	-6	6	(3, 3)
Castelnuovo	6	2	0	0	0	2	-4	10	(3, 3)
Rational	7	3	0	0	0	0	-3	12	(3, 4)
K3	8	5	1	1	0	0	0	24	(4, 4)
Rational	8	4	0	0	0	-2	-2	14	(4, 4)
Rational	8	4	0	0	0	-1	-2	15	(3, 5)
Conic bundle	8	3	0	-1	1	0	-4	0	(2, 6)
Rational	8	3	0	0	0	2	-4	10	(4, 4)
Rational	9	5	0	0	0	-1	-1	13	(3, 6)
Conic bundle	9	4	0	-1	1	-3	-3	3	(3, 6)
Rational	9	5	0	0	0	-3	-1	15	(4, 5)
K3	9	6	1	1	0	-1	1	25	(4, 5)
Enriques	10	6	0	0	0	0	0	12	(3, 7)
Z_A^{10}	10	6	0	0	0	-3	0	15	(4, 6)
Z_B^{10}	10	7	1	1	0	-1	2	25	(4, 6)
Z_C^{10}	10	6	0	0	0	-4	0	16	(5, 5)
Z_D^{10}	10	7	0	0	0	-8	2	20	(5, 5)
Z_E^{10}	10	7	1	1	0	-2	2	26	(5, 5)
Z_F^{10}	10	8	2	2	0	0	4	36	(5, 5)
C_A^{10}	10	4	0	-1	1	-2	-4	2	(5, 5)
C_B^{10}	10	5	0	-1	1	-6	-2	6	(5, 5)
C_C^{10}	10	6	0	-1	1	-10	0	10	(5, 5)

Table 2.1: Table with invariants of non-degenerate congruences of degree at most 10.

Our work is mainly concerned with surfaces S of type Z_E^{10} . These surfaces contain two skew (-1) -lines whose contraction defines a K3 surface of genus 7. We can also describe their ideal sheaves \mathcal{I}_{S/Q^4} via a resolution of known vector bundles, such as the so-called spinor bundles \mathcal{E} and \mathcal{E}' of Q^4 , and the vector bundle $\dot{\mathcal{E}}$ (see [5, §1.3] for its construction). Under the isomorphism $Q^4 \cong G(2,4)$, the spinor bundles \mathcal{E} and \mathcal{E}' correspond to the tautological bundle $\mathcal{U}_{G(2,4)}$ and the dual of the tautological quotient bundle $\mathcal{Q}_{G(2,4)}$, respectively.

2.3.2 Proposition. *Let $S \subseteq Q^4$ be a surface of type Z_B^{10} . Then S is not contained in a quadratic complex and has ideal sheaf resolution*

$$0 \rightarrow \mathcal{E}(-3) \oplus \mathcal{E}(-3) \oplus \mathcal{O}_{Q^4}(-4) \rightarrow \mathcal{O}_{Q^4}(-3)^{\oplus 6} \rightarrow \mathcal{I}_{S/Q^4} \rightarrow 0.$$

Let $S \subseteq Q^4$ be a surface of type Z_E^{10} . If S is contained in a quadratic complex, then it has ideal sheaf resolution

$$0 \rightarrow \mathcal{O}_{Q^4}(-4)^{\oplus 2} \oplus \mathcal{O}_{Q^4}(-3) \rightarrow \dot{\mathcal{E}}(-2) \oplus \mathcal{O}_{Q^4}(-2) \oplus \mathcal{O}_{Q^4}(-4) \rightarrow \mathcal{I}_{S/Q^4} \rightarrow 0.$$

If S is not contained in a quadratic complex, then it has ideal sheaf resolution

$$0 \rightarrow \mathcal{E}(-3) \oplus \mathcal{E}'(-3) \oplus \mathcal{O}_{Q^4}(-4) \rightarrow \mathcal{O}_{Q^4}(-3)^{\oplus 6} \rightarrow \mathcal{I}_{S/Q^4} \rightarrow 0.$$

Both cases occur.

Proof. See [24, Proposition 4.3]. □

In fact, only surfaces of type Z_E^{10} not contained in a quadratic complex will be relevant to us. Thus, we have to make a clear distinction thereof.

2.3.3 Definition. Let $S \subseteq Q^4$ be a surface of type Z_E^{10} . We say that S is of type ${}^I Z_E^{10}$ if $h^0(\mathcal{I}_{S/Q^4}(2)) > 0$, and we say that S is of type ${}^{II} Z_E^{10}$ if $h^0(\mathcal{I}_{S/Q^4}(2)) = 0$.

In other words, S is of type ${}^I Z_E^{10}$ if and only if S is contained in a quadratic complex, and S is of type ${}^{II} Z_E^{10}$ if and only if S is not contained in a quadratic complex. More generally, we say that a surface $S \subseteq \mathbb{P}^5$ is of type ${}^{II} Z_E^{10}$ if there exists a smooth quadric Q^4 containing S and this surface is of type ${}^{II} Z_E^{10}$ according to the previous definition.

Define the polynomial $P(t) := 5t^2 - t + 2 \in \mathbb{Q}[k]$ and consider the Hilbert schemes $\text{Hilb}_{Q^4}^{P(t)}$ and $\text{Hilb}_{\mathbb{P}^5}^{P(t)}$ parametrizing closed subschemes of Q^4 and \mathbb{P}^5 , respectively, with Hilbert polynomial $P(t)$.

2.3.4 Proposition. *Surfaces of type Z_B^{10} and Z_E^{10} are the only classes of smooth surfaces in Q^4 with Hilbert polynomial $P(t)$.*

Proof. Indeed, a smooth surface S with Hilbert polynomial $P(t) = 5t^2 - t + 2 = \frac{H_S^2}{2}t^2 - \frac{H_S K_S}{2}t + \chi(\mathcal{O}_S)$ has degree $H_S^2 = 5 \times 2 = 10$. Also, S must necessarily have sectional genus 7, since $2\pi - 2 = H_S^2 + H_S K_S = 10 + 2 = 12$. By the classification of non-degenerate smooth surfaces of degree 10 in \mathbb{P}^4 by K. Ranestad and S. Popescu (see [44, Theorem 0.16]), no non-degenerate smooth surface of degree 10 in \mathbb{P}^4 can have sectional genus 7. Moreover, a smooth surface of degree 10 contained in \mathbb{P}^3 cannot have sectional genus 7 either, for a general hyperplane section is a smooth plane curve. Hence, S is non-degenerate. Therefore, S must be of type Z_B^{10} , Z_D^{10} , or Z_E^{10} by table 2.1. However, we can verify that only surfaces S of type Z_B^{10} and Z_E^{10} satisfy $\chi(\mathcal{O}_S) = \frac{K_S^2 + c_2(S)}{12} = 2$ using the same table. □

2.3.5 Lemma. *Let S be a surface of type Z_B^{10} . Then*

(a) $\text{Hilb}_{Q^4}^{P(t)}$ is smooth at $[S]$ and the Zariski tangent space $T_{[S]}\text{Hilb}_{Q^4}^{P(t)}$ has dimension 36.

(b) $\text{Hilb}_{\mathbb{P}^5}^{P(t)}$ is smooth at $[S]$ and the Zariski tangent space $T_{[S]}\text{Hilb}_{\mathbb{P}^5}^{P(t)}$ has dimension 56.

Let S be a surface of type ${}^{II}Z_E^{10}$. Then

(c) $\text{Hilb}_{Q^4}^{P(t)}$ is smooth at $[S]$ and the Zariski tangent space $T_{[S]}\text{Hilb}_{Q^4}^{P(t)}$ has dimension 38.

(d) $\text{Hilb}_{\mathbb{P}^5}^{P(t)}$ is smooth at $[S]$ and the Zariski tangent space $T_{[S]}\text{Hilb}_{\mathbb{P}^5}^{P(t)}$ has dimension 58.

Proof. Let $S \subseteq Q^4$ be a surface of type ${}^{II}Z_E^{10}$. We can compute the cohomology groups $H^1(Q^4, N_{S/Q^4})$ and $H^2(Q^4, N_{S/Q^4})$ as $\text{Ext}^2(\mathcal{I}_{S/Q^4}, \mathcal{I}_{S/Q^4})$ and $\text{Ext}^3(\mathcal{I}_{S/Q^4}, \mathcal{I}_{S/Q^4})$, respectively, by [5, Lemma 2.3]. We thus apply the functor $\text{Hom}(-, \mathcal{I}_{S/Q^4})$ to the resolution sequence of \mathcal{I}_{S/Q^4} found in Proposition 2.3.2 to get a long exact sequence of right derived functors which will help us to compute them. Below we have an extract of the long exact sequences with the relevant spaces to consider. We have rewritten it in a cohomological style using well-known properties of the functors $\text{Ext}^i(-, \mathcal{I}_{S/Q^4})$ and identifications $(\mathcal{E}(-3))^* = \mathcal{E}(4)$ and $(\mathcal{E}'(-3))^* = \mathcal{E}'(4)$.

$$\begin{aligned} 0 &\rightarrow H^0(\mathcal{I}_{S/Q^4}(3))^{\oplus 6} \rightarrow H^0((\mathcal{E}(4) \oplus \mathcal{E}'(4)) \otimes \mathcal{I}_{S/Q^4}) \oplus H^0(\mathcal{I}_{S/Q^4}(4)) \rightarrow H^0(N_{S/Q^4}) \\ &\rightarrow H^1(\mathcal{I}_{S/Q^4}(3))^{\oplus 6} \rightarrow H^1((\mathcal{E}(4) \oplus \mathcal{E}'(4)) \otimes \mathcal{I}_{S/Q^4}) \oplus H^1(\mathcal{I}_{S/Q^4}(4)) \rightarrow H^1(N_{S/Q^4}) \\ &\rightarrow H^2(\mathcal{I}_{S/Q^4}(3))^{\oplus 6} \rightarrow H^2((\mathcal{E}(4) \oplus \mathcal{E}'(4)) \otimes \mathcal{I}_{S/Q^4}) \oplus H^2(\mathcal{I}_{S/Q^4}(4)) \rightarrow H^2(N_{S/Q^4}) \\ &\rightarrow H^3(\mathcal{I}_{S/Q^4}(3))^{\oplus 6} \rightarrow H^3((\mathcal{E}(4) \oplus \mathcal{E}'(4)) \otimes \mathcal{I}_{S/Q^4}) \oplus H^3(\mathcal{I}_{S/Q^4}(4)) \rightarrow 0 \end{aligned}$$

It is easy to see that $H^1(\mathcal{I}_{S/Q^4}(3)) = H^1(\mathcal{I}_{S/Q^4}(4)) = 0$ using the resolution sequence of \mathcal{I}_{S/Q^4} . Moreover, we have $H^i(\mathcal{I}_{S/Q^4}(l)) = 0$ for all $l \geq 1$ and all $i \geq 2$ by [24, Proposition 4.3]. Hence, $H^1(N_{S/Q^4}) = H^1(\mathcal{E}(4) \otimes \mathcal{I}_{S/Q^4}) \oplus H^1(\mathcal{E}'(4) \otimes \mathcal{I}_{S/Q^4})$ and $H^2(N_{S/Q^4}) = H^2(\mathcal{E}(4) \otimes \mathcal{I}_{S/Q^4}) \oplus H^2(\mathcal{E}'(4) \otimes \mathcal{I}_{S/Q^4})$.

To compute $H^1(\mathcal{E}(4) \otimes \mathcal{I}_{S/Q^4})$ and $H^2(\mathcal{E}(4) \otimes \mathcal{I}_{S/Q^4})$ we tensor the resolution sequence of \mathcal{I}_{S/Q^4} with the vector bundle $\mathcal{E}(4)$. The short sequence thus obtained remains exact and we can now consider its associated long exact sequence in cohomology

$$\begin{aligned} 0 &\rightarrow H^0(\mathcal{E} \otimes \mathcal{E}^*) \oplus H^0(\mathcal{E} \otimes \mathcal{E}'^*) \oplus H^0(\mathcal{E}) \rightarrow H^0(\mathcal{E}(1))^{\oplus 6} \rightarrow H^0(\mathcal{E}(4) \otimes \mathcal{I}_{S/Q^4}) \\ &\rightarrow H^1(\mathcal{E} \otimes \mathcal{E}^*) \oplus H^1(\mathcal{E} \otimes \mathcal{E}'^*) \oplus H^1(\mathcal{E}) \rightarrow H^1(\mathcal{E}(1))^{\oplus 6} \rightarrow H^1(\mathcal{E}(4) \otimes \mathcal{I}_{S/Q^4}) \\ &\rightarrow H^2(\mathcal{E} \otimes \mathcal{E}^*) \oplus H^2(\mathcal{E} \otimes \mathcal{E}'^*) \oplus H^2(\mathcal{E}) \rightarrow H^2(\mathcal{E}(1))^{\oplus 6} \rightarrow H^2(\mathcal{E}(4) \otimes \mathcal{I}_{S/Q^4}) \\ &\rightarrow H^3(\mathcal{E} \otimes \mathcal{E}^*) \oplus H^3(\mathcal{E} \otimes \mathcal{E}'^*) \oplus H^3(\mathcal{E}) \rightarrow H^3(\mathcal{E}(1))^{\oplus 6} \rightarrow H^3(\mathcal{E}(4) \otimes \mathcal{I}_{S/Q^4}) \rightarrow \dots \end{aligned}$$

We have $H^i(\mathcal{E} \otimes \mathcal{E}^*) \oplus H^i(\mathcal{E} \otimes \mathcal{E}'^*) \oplus H^i(\mathcal{E}) = H^i(\mathcal{E}(1)) = 0$ for $i = 1, 2$, and 3 by [5, §1.4]. Hence, $H^1(\mathcal{E}(4) \otimes \mathcal{I}_{S/Q^4}) = H^2(\mathcal{E}(4) \otimes \mathcal{I}_{S/Q^4}) = 0$. Similarly, we get $H^1(\mathcal{E}'(4) \otimes \mathcal{I}_{S/Q^4}) = H^2(\mathcal{E}'(4) \otimes \mathcal{I}_{S/Q^4}) = 0$. Consequently, $H^1(N_{S/Q^4}) = H^2(N_{S/Q^4}) = 0$. In particular, $\text{Hilb}_{Q^4}^{P(t)}$ is smooth at $[S]$ and $\dim(T_{[S]}\text{Hilb}_{Q^4}^{P(t)}) = \dim(H^0(N_{S/Q^4})) = \chi(N_{S/Q^4}) = 38$ by Corollary 2.1.2.

Now consider the short exact sequence

$$0 \rightarrow N_{S/Q^4} \rightarrow N_{S/\mathbb{P}^5} \rightarrow N_{Q^4/\mathbb{P}^5}|_S = \mathcal{O}_S(2) \rightarrow 0$$

induced from the nested sequence of inclusions of smooth varieties $S \subseteq Q^4 \subseteq \mathbb{P}^5$ and also the tautological short exact sequence

$$0 \rightarrow \mathcal{I}_{S/Q^4} \rightarrow \mathcal{O}_{Q^4} \rightarrow \mathcal{O}_S \rightarrow 0.$$

The long exact sequences in cohomology induced from the two sequences above allow us to compute easily $H^0(N_{S/\mathbb{P}^5}) = H^0(N_{S/Q^4}) \oplus \mathbb{C}^{20}$ and $H^1(N_{S/\mathbb{P}^5}) = H^2(N_{S/\mathbb{P}^5}) = 0$. Hence, $\text{Hilb}_{\mathbb{P}^5}^{P(t)}$ is also smooth at $[S]$ and $\dim(T_{[S]}\text{Hilb}_{\mathbb{P}^5}^{P(t)}) = \dim(H^0(N_{S/\mathbb{P}^5})) = 58$.

Finally, if S is a surface of type Z_B^{10} , the same arguments can be applied to prove that the Hilbert schemes are smooth at $[S]$ and the corresponding Zariski tangent spaces have the dimensions stated in the Lemma. \square

2.3.6 Proposition. *There is a smooth open subset $U_{Q^4} \subseteq \text{Hilb}_{Q^4}^{P(t)}$ of dimension 38 that parametrizes surfaces of type ${}^{II}Z_E^{10}$ in Q^4 . There is a smooth open subset $U \subseteq \text{Hilb}_{\mathbb{P}^5}^{P(t)}$ of dimension 58 that parametrizes surfaces of type ${}^{II}Z_E^{10}$ in \mathbb{P}^5 .*

Proof. The only smooth surfaces parametrized by $\text{Hilb}_{Q^4}^{P(t)}$ are those of type Z_B^{10} and Z_E^{10} by Proposition 2.3.4. By the semi-continuity theorem, there is a open subset of $\text{Hilb}_{Q^4}^{P(t)}$ characterized by the equation $h^0(\mathcal{I}_{S/Q^4}(2)) = 0$. Observe that surfaces of type Z_B^{10} and ${}^{II}Z_E^{10}$ are contained in this open subset, while surfaces of type ${}^I Z_E^{10}$ are not. Since the locus of smooth surfaces in $\text{Hilb}_{Q^4}^{P(t)}$ is open, and the tangent space dimensions differ from one class of surfaces to the other, we see that there are two disjoint smooth open subsets of $\text{Hilb}_{Q^4}^{P(t)}$ parametrizing surfaces of type Z_B^{10} and ${}^{II}Z_E^{10}$, respectively. We thus have a smooth open subset U_{Q^4} of $\text{Hilb}_{Q^4}^{P(t)}$ of dimension 38 parametrizing surfaces of type ${}^{II}Z_E^{10}$ by Lemma 2.3.5.

Let $[S] \in \text{Hilb}_{\mathbb{P}^5}^{P(t)}$ and consider the short exact sequence

$$0 \rightarrow \mathcal{I}_{S/\mathbb{P}^5}(2) \rightarrow \mathcal{O}_{\mathbb{P}^5}(2) \rightarrow \mathcal{O}_S(2) \rightarrow 0.$$

By the additivity property of the Euler characteristic χ over exact sequences, we have $\chi(\mathcal{I}_{S/\mathbb{P}^5}(2)) + P(2) = \chi(\mathcal{O}_{\mathbb{P}^5}(2))$, or equivalently, $\chi(\mathcal{I}_{S/\mathbb{P}^5}(2)) = 1$. In particular, the inequality $h^0(\mathcal{I}_{S/\mathbb{P}^5}(2)) + h^2(\mathcal{I}_{S/\mathbb{P}^5}(2)) \geq 1$ holds. On the other hand, we have $h^2(\mathcal{I}_{S/\mathbb{P}^5}(2)) = 0$ if the surface S is of type Z_B^{10} or Z_E^{10} (see the proof of [24, Proposition 4.3]). By the semi-continuity theorem, there is an open subset of $W \subseteq \text{Hilb}_{\mathbb{P}^5}^{P(t)}$ containing surfaces of type Z_B^{10} and Z_E^{10} and characterized by the condition $h^2(\mathcal{I}_{S/\mathbb{P}^5}(2)) = 0$. In particular, $h^0(\mathcal{I}_{S/\mathbb{P}^5}(2)) \geq 1$ on W , and thus we have $W = \bigcup_{V(2) \in |\mathcal{O}_{\mathbb{P}^5}(2)|} \text{Hilb}_{V(2)}^{P(t)}$. Now consider the open subset V of W characterized by the equation $h^0(\mathcal{I}_{S/\mathbb{P}^5}(2)) = 1$. The preimage of the locus of smooth quadrics in $|\mathcal{O}_{\mathbb{P}^5}(2)|$ under the canonical morphism $V \rightarrow |\mathcal{O}_{\mathbb{P}^5}(2)|$ is open in V . Moreover, the locus of smooth surfaces in $\text{Hilb}_{\mathbb{P}^5}^{P(t)}$ is open as well. Hence, there is an open subset of $\text{Hilb}_{\mathbb{P}^5}^{P(t)}$ parametrizing smooth surfaces contained in a unique smooth quadric. These surfaces can only be of type Z_B^{10} and ${}^{II}Z_E^{10}$ by Proposition 2.3.4. Since the tangent space dimensions differ from one class of surfaces to the other, we see that there are two disjoint smooth open subsets of $\text{Hilb}_{\mathbb{P}^5}^{P(t)}$ parametrizing surfaces of type Z_B^{10} and ${}^{II}Z_E^{10}$, respectively. We thus have a smooth open subset U of $\text{Hilb}_{\mathbb{P}^5}^{P(t)}$ of dimension 58 parametrizing surfaces of type ${}^{II}Z_E^{10}$ by Lemma 2.3.5. \square

2.3.7 Proposition. *There is a rational dominant map $\mathbb{P} \dashrightarrow U_{Q^4}$, where \mathbb{P} is a projective space. In particular, U_{Q^4} is unirational.*

Proof. We have $\text{Hom}(\mathcal{E} \oplus \mathcal{E}' \oplus \mathcal{O}_{Q^4}(-1), \mathcal{O}_{Q^4}^{\oplus 6}) = (V_6^* \otimes V_4^*) \oplus (V_6^* \otimes V_4) \oplus (V_6^* \otimes \wedge^2 V_4^*)$, where we set $V_4^* = H^0(Q^4, \mathcal{E}^*)$, $V_4 = H^0(Q^4, \mathcal{E}'^*)$, $\wedge^2 V_4^* = H^0(Q^4, \mathcal{O}_{Q^4}(1))$, $V_6^* = H^0(Q^4, \mathcal{O}_{Q^4}^{\oplus 6})$ for vector spaces of dimensions 4 and 6 denoted by V_4 and V_6 , respectively. In other words, three tensors define a morphism $\mathcal{E} \oplus \mathcal{E}' \oplus \mathcal{O}_{Q^4}(-1) \rightarrow \mathcal{O}_{Q^4}^{\oplus 6}$. If the tensors are general, then the rank of the induced morphism drops in codimension two over a surface of type Z_E^{10} by [10, Proposition 6.3]. Conversely, for each surface S of type ${}^{II}Z_E^{10}$ we have an associated exact sequence

$$0 \rightarrow \mathcal{E} \oplus \mathcal{E}' \oplus \mathcal{O}_{Q^4}(-1) \rightarrow \mathcal{O}_{Q^4}^{\oplus 6} \rightarrow \mathcal{I}_{S/Q^4}(3) \rightarrow 0$$

by Proposition 2.3.2. Therefore, an open subscheme of the projective space $\mathbb{P} := \mathbb{P}((V_6^* \otimes V_4) \oplus (V_6^* \otimes \wedge^2 V_4^*) \oplus (V_6^* \otimes V_4^*))$ maps onto the open subscheme U_{Q^4} of $\text{Hilb}_{Q^4}^{P(t)}$. This defines a dominant rational map $\mathbb{P} \dashrightarrow U_{Q^4}$, and thus unirationality is proved. \square

Let $\mathcal{U}_{Q^4} := \frac{U_{Q^4}}{\text{Aut}(Q^4)}$ be the coarse moduli space of surfaces of type ${}^{II}Z_E^{10}$ in Q^4 .

2.3.8 Proposition. *Let $\mathcal{F}_7^{[2]}$ be the universal Hilbert scheme of zero-dimensional schemes of length 2 in a polarized K3 surface of genus 7. Then there is an open immersion $\mathcal{U}_{Q^4} \hookrightarrow \mathcal{F}_7^{[2]}$. In particular, the minimal model of a very general surface S of type ${}^{II}Z_E^{10}$ has Picard rank 1.*

Proof. The locus of triples $(S_0, p, p') \in \mathcal{F}_7^{[2]}$, where p and p' are distinct points on a K3 surface S_0 of genus 7 polarized by H_{S_0} , and such that the linear system $|H_{S_0} - E_p - E_{p'}|$ embeds $Bl_{p,p'}(S_0)$ into \mathbb{P}^5 , is an open set V , since very ampleness is an open condition. Observe that such surfaces in \mathbb{P}^5 have Hilbert polynomial $P(t) = 5t^2 - t + 2$.

There exists a smooth open subset $U \subseteq \text{Hilb}_{\mathbb{P}^5}^{P(t)}$ of dimension 58 parametrizing surfaces of type ${}^{II}Z_E^{10}$ in \mathbb{P}^5 by Proposition 2.3.6. This open subset is clearly invariant under the action of $PGL(6)$. Moreover, the moduli spaces $\mathcal{U}_{Q^4} = U_{Q^4}/\text{Aut}(Q^4)$ and $U/PGL(6)$ are isomorphic. Hence, the moduli space $\text{Hilb}_{\mathbb{P}^5}^{P(t)}/PGL(6)$ contains \mathcal{U}_{Q^4} as an open subset. In particular, the morphism $\mu : V \rightarrow \text{Hilb}_{\mathbb{P}^5}^{P(t)}/PGL(6)$ given by $(S_0, p, p') \mapsto [Bl_{p,p'}(S_0) \hookrightarrow \mathbb{P}^5]$ can be restricted to the non-empty¹ open set $\mu^{-1}(\mathcal{U}_{Q^4}) \subseteq V \subseteq \mathcal{F}_7^{[2]}$.

On the other hand, if S is a surface of type ${}^{II}Z_E^{10}$, then the linear system $|H_S + K_S|$ is base-point free by [51, Proposition 2.2]. It is easy to see that this linear system induces a morphism $S \rightarrow \mathbb{P}^7$ that realizes the blow-up map of a K3 surface $S_0 \subseteq \mathbb{P}^7$ of degree 12 at two points p and p' . This allows us to define the morphism $\nu : \mathcal{U}_{Q^4} \rightarrow \mathcal{F}_7^{[2]}$ by $[S] \mapsto (S_0, p, p')$. Evidently, μ and ν are inverse to each other. Hence, $\mathcal{U}_{Q^4} \cong \mu^{-1}(\mathcal{U}_{Q^4}) \subseteq \mathcal{F}_7^{[2]}$. The last claim follows from the well-known fact that a very general element of the moduli space of polarized K3 surfaces of genus 7 has Picard rank 1. \square

2.3.9 Remark. Proposition 2.3.8 also proves that \mathcal{U}_{Q^4} is unirational by [20, Theorem 5.1].

¹See Theorem 2.4.3 for an explicit example of a K3 surface that is projected onto a surface of type ${}^{II}Z_E^{10}$.

2.4 Special cubic fourfolds of discriminant 14

A special cubic fourfold is a smooth cubic hypersurface $X \subseteq \mathbb{P}^5$ containing a surface S not homologous to a complete intersection. The saturation of the rank 2 sublattice of $H^4(X, \mathbb{Z})$ generated by the square of the hyperplane class h of X and the surface S has discriminant $d = 3S^2 - \deg(S)^2$. We also say that X has discriminant d .

The coarse moduli space $\mathcal{C} = \frac{\mathbf{sm}|\mathcal{O}_{\mathbb{P}^5}(3)|}{PGL(6)}$ of smooth cubic fourfolds in \mathbb{P}^5 is quasi-projective and 20-dimensional. Special cubic fourfolds in \mathcal{C} are parametrized by a countable family of irreducible divisors \mathcal{C}_d , called Hassett divisors, indexed by discriminants $d > 6$ and $d = 0$ or $2 \pmod 6$ by [27, Theorem 4.3.1].

Moreover, a special cubic fourfold X in \mathcal{C}_d has a Hodge-theoretically associated degree- d polarized K3 surface (T, L) , in the sense that there exists a Hodge isometry $H^4(X, \mathbb{Z}) \supseteq \langle h^2, S \rangle^\perp \cong L^\perp \subseteq H^2(T, \mathbb{Z})(-1)$, if and only if d is not divisible by 4, 9, or any odd prime $p = 2 \pmod 3$ by [27, Theorem 5.1.3]. We say that such integers d are admissible.

Although it is expected that the very general smooth cubic fourfold is non-rational, no example of a provable non-rational cubic fourfold has been found as of this day. In fact, the famous Kuznetsov conjecture asserts that special cubic fourfolds with an associated K3 surface are exactly the rational cubic fourfolds in \mathcal{C} . Some special cubic fourfolds with an associated K3 surface have been shown to be rational, thus supporting the veracity of the conjecture. For example, cubics parametrized by the Hassett divisors \mathcal{C}_{14} , \mathcal{C}_{26} , \mathcal{C}_{38} , and \mathcal{C}_{42} are known to be rational (see [11], [47], and [48]). Observe that the first admissible discriminants d are precisely 14, 26, 38, and 42.

It is well known that the closure of the locus of Pfaffian cubics, i.e., smooth cubics defined by the Pfaffian of a 6×6 skew-symmetric matrix of linear forms in \mathbb{P}^5 , is the Hassett divisor \mathcal{C}_{14} . In fact, a smooth cubic fourfold is Pfaffian if and only if it contains a quintic del Pezzo surface by [7, Proposition 9.2]. Moreover, \mathcal{C}_{14} can also be characterized as the closure of the locus of smooth cubic fourfolds containing a rational normal quartic scroll. The rich geometry of cubics in \mathcal{C}_{14} is classically tied to the existence of the quartic scrolls and quintic del Pezzo surfaces. Therefore, it is important to investigate the existence of other surfaces characterizing a general cubic of discriminant 14.

We aim to show that surfaces of type ${}^{II}Z_E^{10}$ also characterize cubics in the Hassett divisor \mathcal{C}_{14} by following the approach of H. Nuer in [39], where he used deformation theory for Hilbert schemes, semicontinuity arguments, and Macaulay2 computations to prove the following theorem.

2.4.1 Theorem. *The generic element of \mathcal{C}_d for $12 \leq d \leq 38$ contains a smooth rational surface obtained as the blow-up of \mathbb{P}^2 at p generic points and embedded into \mathbb{P}^5 via a very ample linear system*

$$|H| = |aL - (E_1 + \dots + E_i) - 2(E_{i+1} + \dots + E_{i+j}) - 3(E_{i+j+1} + \dots + E_p)|, \text{ where } a \in \mathbb{Z}.$$

If $d = 44$, the generic element of \mathcal{C}_{44} contains a Fano model of an Enriques surface.

Proof. See [39, Theorem 3.1 and Theorem 3.2]. □

First of all, we need to make sense of our claim by proving

2.4.2 Lemma. *If S is a surface of type ${}^{II}Z_E^{10}$ and X is a smooth cubic fourfold that contains S , then X is special of discriminant 14, i.e. $X \in \mathcal{C}_{14}$.*

Proof. The surface S is not homologous to a complete intersection, since $S^2 = c_2(N_{S/X}) = 6\deg(S) + 3hK_S + K_S^2 - c_2(S) = (6 \times 10) + (3 \times 2) - 2 - 26 = 38$ according to [27, §4] and table 2.1, but 38 is not divisible by $(h^2)^2 = 3$. Moreover, $\langle h^2, S \rangle$ is a sublattice of $H^4(X, \mathbb{Z})$ of rank 2 and discriminant $3S^2 - \deg(S)^2 = (3 \times 38) - 10^2 = 114 - 100 = 14$ that is necessarily saturated, since 14 is a square-free integer. \square

2.4.3 Theorem. *A general cubic $X \in \mathcal{C}_{14}$ contains a surface of type ${}^{II}Z_E^{10}$.*

Proof. Consider the Hilbert scheme of flags

$$\mathcal{FH} := \{[S \subseteq X] : [S] \in \text{Hilb}_{\mathbb{P}^5}^{P(t)} \text{ and } [X] \in \mathbf{sm}|\mathcal{O}_{\mathbb{P}^5}(3)|\} \subseteq \text{Hilb}_{\mathbb{P}^5}^{P(t)} \times \mathbf{sm}|\mathcal{O}_{\mathbb{P}^5}(3)|,$$

where $P(t) := 5t^2 - t + 2$. Note that the first projection $p : \mathcal{FH} \rightarrow \text{Hilb}_{\mathbb{P}^5}^{P(t)}$ has fiber $p^{-1}([S]) = \mathbb{P}(H^0(\mathcal{I}_{S/\mathbb{P}^5}(3)))$, while the second projection $q : \mathcal{FH} \rightarrow \mathbf{sm}|\mathcal{O}_{\mathbb{P}^5}(3)|$ has fiber $q^{-1}([X]) = \text{Hilb}_X^{P(t)}$.

Let $[S] \in \text{Hilb}_{\mathbb{P}^5}^{P(t)}$. We have the short exact sequence

$$0 \rightarrow \mathcal{I}_{S/\mathbb{P}^5}(3) \rightarrow \mathcal{O}_{\mathbb{P}^5}(3) \rightarrow \mathcal{O}_S(3) \rightarrow 0.$$

By the additivity property of the Euler characteristic χ over exact sequences, we have $\chi(\mathcal{I}_{S/\mathbb{P}^5}(3)) + P(3) = \chi(\mathcal{O}_{\mathbb{P}^5}(3))$, or equivalently, $\chi(\mathcal{I}_{S/\mathbb{P}^5}(3)) = 12$. In particular, the inequality $h^0(\mathcal{I}_{S/\mathbb{P}^5}(3)) + h^2(\mathcal{I}_{S/\mathbb{P}^5}(3)) \geq 12$ holds. On the other hand, we have $h^2(\mathcal{I}_{S/\mathbb{P}^5}(3)) = 0$ for any surface S of type Z_E^{10} (see the proof of [24, Proposition 4.3]). In particular, we have $h^0(\mathcal{I}_{S/\mathbb{P}^5}(3)) \geq 12$ for all surfaces S of type Z_E^{10} . In fact, we have $h^0(\mathcal{I}_{S/\mathbb{P}^5}(3)) = 12$ for surfaces S of type ${}^{II}Z_E^{10}$, since $h^0(\mathcal{I}_{S/\mathbb{P}^5}(1)) = 0$, $h^0(\mathcal{I}_{S/\mathbb{P}^5}(2)) = 1$ and $h^0(\mathcal{I}_{S/Q^4}(3)) = 6$ by Proposition 2.3.2.

Recall that there exists a smooth open set $U \subseteq \text{Hilb}_{\mathbb{P}^5}^{P(t)}$ of dimension 58 that parametrizes surfaces of type ${}^{II}Z_E^{10}$ in \mathbb{P}^5 by Proposition 2.3.6. Then the preimage $p^{-1}(U) \subseteq \mathcal{FH}$ is a smooth projective bundle over U . Therefore,

$$\dim_{[S \subseteq X]}(p^{-1}(U)) = \dim(T_{[S]} \text{Hilb}_{\mathbb{P}^5}^{P(t)}) + h^0(\mathcal{I}_{S/\mathbb{P}^5}(3)) - 1 = 58 + 12 - 1 = 69$$

for any flag $[S \subseteq X] \in p^{-1}(U)$.

Observe that $q : \mathcal{FH} \rightarrow \mathbf{sm}|\mathcal{O}_{\mathbb{P}^5}(3)|$ maps $p^{-1}(U)$ into the preimage of the Hassett divisor \mathcal{C}_{14} under the quotient map $\mathbf{sm}|\mathcal{O}_{\mathbb{P}^5}(3)| \rightarrow \mathcal{C} = \mathbf{sm}|\mathcal{O}_{\mathbb{P}^5}(3)|/PGL(6)$ by Lemma 2.4.2. Thus, $\dim(q(p^{-1}(U))) \leq \dim(\mathcal{C}_{14}) + \dim PGL(6) = 54$. By applying the theorem on the dimension of the fibers, we have

$$\begin{aligned} h^0(N_{S/X}) &= \dim(T_{[S]} \text{Hilb}_X^{P(t)}) \\ &\geq \dim_{[S \subseteq X]}(q^{-1}([X])) \\ &\geq \dim(p^{-1}(U)) - \dim(q(p^{-1}(U))) \\ &\geq \dim(p^{-1}(U)) - 54 \\ &= 69 - 54 \\ &= 15 \end{aligned}$$

for any flag $[S \subseteq X] \in q^{-1}([X])$.

We can use Macaulay2 to construct an explicit surface S of type $II Z_E^{10}$ and find a smooth cubic X containing S such that $h^0(N_{S/X}) = 15$:

Macaulay2, version 1.22.0.1

with packages: ConwayPolynomials, Elimination, IntegralClosure, InverseSystems, Isomorphism, LLLBases, MinimalPrimes, OnlineLookup, PrimaryDecomposition, ReesAlgebra, Saturation, TangentCone, Varieties

```
i1 : needsPackage "K3Surfaces"

o1 = K3Surfaces

o1 : Package

i2 : S0 = K3(7,5,0)

o2 = K3 surface with rank 2 lattice defined by the intersection matrix:
                                     | 12 5 |
                                     | 5  0 |
-- (1,0): K3 surface of genus 7 and degree 12 containing elliptic curve of degree 5

o2 : Lattice-polarized K3 surface

i3 : S = project({1,1},S0(1,0))
-- *** simulation ***
-- surface of degree 12 and sectional genus 7 in PP^7 (quadrics: 10, cubics: 64)
-- surface of degree 11 and sectional genus 7 in PP^6 (quadrics: 5, cubics: 34)
-- surface of degree 10 and sectional genus 7 in PP^5 (quadrics: 1, cubics: 12)

-- (degree and genus are as expected)

o3 = S

o3 : ProjectiveVariety, surface in PP^5

i4 : singularLocus S

o4 = empty subscheme of PP^5

o4 : ProjectiveVariety, empty subscheme of PP^5

i5 : euler S

o5 = 26

i6 : Q = random(2,S)

o6 = Q

o6 : ProjectiveVariety, hypersurface in PP^5

i7 : describe Q

o7 = ambient:..... PP^5
     dim:..... 4
     codim:..... 1
     degree:..... 2
     generators:..... 2^1
     purity:..... true
```

```

dim sing. 1.:..... -1

i8 : X = random(3,S)

o8 = X

o8 : ProjectiveVariety, hypersurface in PP^5

i9 : describe X

o9 = ambient:..... PP^5
dim:..... 4
codim:..... 1
degree:..... 3
generators:..... 3^1
purity:..... true
dim sing. 1.:..... -1

i10 : rank HH^0(sheaf Hom((ideal S/ideal X)**(ring(ideal S)/ideal S),ring(ideal S)/ideal S))

o10 = 15

```

This implies that $\dim(q(p^{-1}(U))) = 54$. Thus, $\overline{q(p^{-1}(U))}$ fills the preimage of \mathcal{C}_{14} and the result follows. \square

2.5 A note on surfaces of type ${}^{II}Z_E^{10}$ and quintic Del Pezzo surfaces

If X is a special cubic fourfold of discriminant 14 containing a quartic scroll Σ and a quintic Del Pezzo surface D , then it is easy to see that $\Sigma = 3h^2 - D$ in $H^4(X, \mathbb{Z})$. In fact, there exists a Segre threefold $\mathbb{P}^1 \times \mathbb{P}^2 \subseteq \mathbb{P}^5$, which has degree 3, that satisfies $(\mathbb{P}^1 \times \mathbb{P}^2) \cap X = \Sigma \cup D$ by [58]. The relation between surfaces of type ${}^{II}Z_E^{10}$ and quintic Del Pezzo surfaces is numerically similar to the case of quartic scrolls and quintic Del Pezzo surfaces. However, we can show that these surfaces cannot be residual to each other via some smooth threefold of \mathbb{P}^5 .

2.5.1 Theorem. *Let X be a smooth cubic fourfold containing a surface S of type ${}^{II}Z_E^{10}$ and a quintic Del Pezzo surface D . Then $S = 5h^2 - D$ in $H^4(X, \mathbb{Z})$, but no smooth quintic threefold $Y \subseteq \mathbb{P}^5$ satisfies $Y \cap X = S \cup D$.*

Proof. There exists an isometry γ of $H^4(X, \mathbb{Z})$ preserving h^2 such that $\langle h^2, D \rangle = \gamma(\langle h^2, S \rangle)$ by [27, Proposition 3.2.4]. In particular, γ defines an integral invertible matrix of the form $\begin{pmatrix} 1 & a \\ 0 & b \end{pmatrix}$ with respect to the bases (h^2, S) and (h^2, D) . In other words, we have $S = ah^2 + bD$ in $H^4(X, \mathbb{Z})$, where $b = \pm 1$ and $a \in \mathbb{Z}$. Then $10 = Sh^2 = 3a + 5b$, and thus we see that $b = -1$ and $a = 5$. Hence, $S = 5h^2 - D$ in $H^4(X, \mathbb{Z})$.

Note that a threefold Y as in the statement must be non-degenerate, since S is. By [41, Proposition 1.5], a non-degenerate smooth quintic threefold $Y \subseteq \mathbb{P}^5$ is a so-called Castelnuovo threefold. This is a quadric bundle over \mathbb{P}^1 with 15 singular fibers. Moreover, its ideal sheaf fits in the short exact sequence

$$0 \rightarrow \mathcal{O}_{\mathbb{P}^5}(-4)^{\oplus 2} \rightarrow \mathcal{O}_{\mathbb{P}^5}(-3)^{\oplus 2} \oplus \mathcal{O}_{\mathbb{P}^5}(-2) \rightarrow \mathcal{I}_{Y/\mathbb{P}^5} \rightarrow 0.$$

From this resolution it is clear that Y is $(2, 3)$ -linked to a \mathbb{P}^3 , e.g., there exist quadric and cubic hypersurfaces Z and X' in \mathbb{P}^5 , respectively, such that $Z \cap X' = Y \cup \mathbb{P}^3$. In particular, both Z and X' are necessarily singular. Since a surface S of type ${}^{II}Z_E^{10}$ is contained in a single quadric hypersurface of \mathbb{P}^5 , and this quadric is smooth, we see that S cannot be contained in Y . In particular, S and D are never linked in X via a smooth quintic Y . \square

3. K3 surfaces and Fano fourfolds of K3 type

In this chapter we study K3 surfaces appearing in the construction of some families of Fano fourfolds of K3 type.

3.1 Introduction

Let $H = \bigoplus_{p+q=k} H^{p,q}$ be a non-trivial Hodge structure of weight k . The level $\lambda(H)$ of H is

$$\lambda(H) := \max\{q - p \mid H^{p,q} \neq 0\}.$$

We say that H is of K3 type if

- (a) $\lambda(H) = 2$;
- (b) $h^{\frac{k-2}{2}, \frac{k+2}{2}} = 1$.

Let X be a smooth Fano variety. We say that X is a Fano manifold of K3 type if $H^*(X, \mathbb{C})$ contains at least one sub-Hodge structure of K3 type, and for every k the Hodge structure of $H^k(X, \mathbb{C})$ has at most one structure of K3 type. When X is a fourfold, this simplifies to the condition $h^{3,1}(X) = 1$.

The classification of families of Fano fourfolds has been completed for index > 1 . We have 35 families. Among them, there exist only three examples of families of Fano fourfolds of K3 type. These are the cubic fourfolds, the Gusein-Mukai fourfolds, and the Verra fourfolds.

Fano fourfolds of index 1 are far from being classified. There are however, some results that pave the way towards a future classification. For example, O. K uchle classified Fano fourfolds of index 1 and second Betti number 1 that can be obtained as the zero locus of direct sums of globally generated irreducible homogeneous vector bundles defined on Grassmannians in [35, 1995]. In his list, we can find only three families of Fano fourfolds of K3 type.

In the recent paper [10, 2021] by M. Bernardara, E. Fatighenti, L. Manivel, and F. Tantarri, a list of 64 new families of Fano fourfolds of K3 type is provided and their geometry is described. Each member from these families of Fano fourfolds is defined as the zero locus of a general section of a completely reducible homogeneous vector bundle defined on a product of flag manifolds. In many cases, the fourfolds from these families admit birational contractions that are realized as blow-ups of Fano varieties along non-minimal K3 surfaces. These families are labeled as K3-N, where N is a two-digit integer.

It is remarkable that for many of these families it is possible to show that the K3 surfaces are related to each other via their transcendental lattices in a geometric way. We abstract this property and define the concept of monoidal Fourier-Mukai partners in consequence:

3.1.1 Definition. We say that two K3 surfaces S_0 and T_0 are monoidal Fourier-Mukai partners, if there exist varieties S, T, V, W, X , and a diagram

$$\begin{array}{ccccc}
 & & X & & \\
 & \swarrow \sigma & & \searrow \tau & \\
 S & \longleftrightarrow & V & & W \longleftrightarrow T
 \end{array}$$

such that the following conditions are satisfied:

- (a) V and W are smooth fourfolds;
- (b) S and T are smooth surfaces;
- (c) The morphism $\sigma : X \rightarrow V$ (resp. $\tau : X \rightarrow W$) is the blow-up of V (resp. W) along S (resp. T);
- (d) The minimal model of S (resp. T) is S_0 (resp. T_0);
- (e) The transcendental lattices of S_0 and T_0 are identified via the blow-up formulae in singular cohomology for σ and τ .

We say that the diagram above is a monoidal FM-diagram of K3 surfaces, and that it is trivial, if additionally $S_0 \cong T_0$.

Clearly, if S_0 and T_0 are monoidal Fourier-Mukai partners, then they derived-equivalent K3 surfaces. In particular, both surfaces have the same Picard rank.

A natural question we can ask is: *which examples of monoidal FM-diagrams of K3 surfaces appearing in [10] are trivial, and which are not?*

We answer this question for the families of Fano fourfolds K3-N, where $N = 32, 33, 36, 37, 39, 41, 46, 47, 48$, and 55 .

Clearly, the problem of determining the triviality of monoidal FM-diagram of K3 surfaces is entirely lattice-theoretical, although the techniques change depending on the Picard ranks of the K3 surfaces. The most interesting cases are the non-trivial monoidal FM-diagrams of K3 surfaces induced by the family of fourfolds K3-32 and K3-33. To show that the K3 surfaces appearing in these examples are non-isomorphic, we adapt a beautiful lattice-theoretic argument of B. Hassett and K.W. Lai from their paper [28] that takes advantage of the geometric structure of the monoidal FM-diagram to compare the discriminant groups of the Néron-Severi lattices of the K3 surfaces. The rest of the families are studied following a well-delimited strategy: in each case we describe explicitly the Néron-Severi lattice N of one of the K3 surfaces and we prove that **(1)** the genus $\mathcal{G}(N)$ is trivial and **(2)** the natural morphism $O(N) \rightarrow O(dN, q_N)$ is surjective. This is enough to conclude that no non-trivial Fourier-Mukai partners of the K3 surface under study can exist, thus implying that the K3 surfaces involved are isomorphic.

Recall that for a lattice N of any rank, conditions **(1)** and **(2)** follow immediately if the inequality $l(N) + 2 \leq rk(N)$ holds. Here $l(N)$ denote the minimal number of generators of the discriminant group dN . However, if N has rank 2, this cannot happen unless N is unimodular. The Néron-Severi lattice of the K3 surfaces of Picard rank 2 under study are non-unimodular, but they are either associated to an indefinite binary quadratic form, or they are elliptic. The computations presented in the first chapter allow us to prove directly both conditions.

3.2 Picard rank 1 cases

3.2.1 The family of Fano fourfolds K3-32

Construction, notation, and conventions.

3.2.1 Theorem. *Let $X = \mathcal{L}(Fl(1, 2; 5), \mathcal{O}(1, 1) \oplus \mathcal{O}(0, 1)^{\oplus 2})^1$. Then the projection $\sigma : X \rightarrow G(1, 5)$ is realized as the blow-up of $G(1, 5)$ along the blow-up of a K3 surface of genus 7 at one point, while the projection $\tau : X \rightarrow G(2, 5)$ is realized as a blow-up of a smooth co-dimension two linear section of $G(2, 5)$ along a K3 surface of genus 7. The corresponding K3 surfaces are derived-equivalent.*

Proof. See [10, Fano K3-32]. □

Any member X of the family of fourfolds constructed in the previous theorem is called Fano fourfold K3-32.

The image of the second projection $\tau : X \rightarrow G(2, 5)$ is a smooth co-dimension two linear section W of the Grassmannian $G(2, 5)$ considered as a subvariety of \mathbb{P}^9 via the Plücker embedding. It is a Fano fourfold of index 3 and degree 5. In fact, it is known that a Fano fourfold of index 3 and degree 5 is unique up to isomorphism and can be realized as a linear section of $G(2, 5)$ by two general hyperplanes. On the other hand, the first projection $\sigma : X \rightarrow G(1, 5)$ is surjective. For convenience, we shall identify the Grassmannian $G(1, 5)$ with projective space \mathbb{P}^4 . Let $S \subseteq \mathbb{P}^4$ (resp. $T \subseteq W$) be the blow-up center of σ (resp. τ). We denote by δ the unique (-1) -line contained in S . Also, write S_0 for the minimal model of S .

We fix notation by means of the following diagram:

$$\begin{array}{ccccccc}
 & & E & \xrightarrow{j_1} & X & \xleftarrow{j_2} & E' \\
 & \swarrow \sigma_1 & & & \swarrow \sigma & & \searrow \tau \\
 \delta & \longrightarrow & S & \xrightarrow{\quad} & \mathbb{P}^4 & & W \xleftarrow{\quad} T \\
 & & & & & & \swarrow \tau_1
 \end{array}$$

The diagram above is a monoidal FM-diagram between S_0 and T . We will show that it is non-trivial. In other words, we will show that S_0 is not isomorphic to T . To do this we shall assume from now on that S_0 , or equivalently, T has Picard rank 1.

We start by applying the well-known blow-up formula for singular cohomology to X to get a composition of Hodge isometries:

$$H^4(\mathbb{P}^4, \mathbb{Z}) \oplus H^2(S, \mathbb{Z}) \xrightarrow{\sigma^* + j_{1*}\sigma_1^*} H^4(X, \mathbb{Z}) \xleftarrow{\tau^* + j_{2*}\tau_1^*} H^4(W, \mathbb{Z}) \oplus H^2(T, \mathbb{Z})$$

The middle cohomology group $H^4(\mathbb{P}^4, \mathbb{Z})$ is freely generated by H^2 , where H is the hyperplane class in \mathbb{P}^4 . In particular, S has class $8H^2$. Concerning the structure of the middle cohomology group of W , we have the following

3.2.2 Proposition. *Let W be a smooth Fano fourfold of index 3 and degree 5. Then $H^4(W, \mathbb{Z})$ is freely generated by the classes of two planes, Ξ and Π . The intersection theory*

¹If E is a globally generated vector bundle on a manifold Y , we denote by $X = \mathcal{L}(Y, E)$ the zero locus of a general section of E in Y . We shall assume that X is a smooth subvariety of Y of codimension $rk(E)$.

of W is given by

$$\Xi^2 = 2$$

$$\Pi^2 = 1$$

$$\Xi\Pi = -1$$

Moreover, $c_2(W) = 9\Xi + 13\Pi$.

Proof. See [45, Corollary 4.7] or [46, Lemma 3.4]. \square

Let H' be the hyperplane class of $G(2, 5) \subseteq \mathbb{P}^9$. Then $\text{Pic}(W) = \mathbb{Z}H'$ by the Lefschetz hyperplane section theorem. In particular, $-K_W = 3H'$.

3.2.3 Remark. It is easy to show that H'^2 is a del Pezzo surface of degree 5 with class $2\Xi + 3\Pi$, while the class of T is given by $5\Xi + 7\Pi$.

We denote by H (resp. H') the pullback $\sigma^*(H)$ (resp. $\tau^*(H')$) on X . We use the same symbol to denote a codimension 2 cycle in X coming from S (resp. T) via $j_{1*}\sigma_1^*$ (resp. $j_{2*}\tau_1^*$). For example, $j_{1*}\sigma_1^*(\delta)$ is denoted simply by δ . With this in mind, we have an identification $H^4(\mathbb{P}^4, \mathbb{Z}) \oplus H^2(S, \mathbb{Z}) = H^4(X, \mathbb{Z}) = H^4(W, \mathbb{Z}) \oplus H^2(T, \mathbb{Z})$. Observe that this lattice is unimodular, although it is not even.

Let H_S (resp. H_T) be the polarization of S (resp. T). Then the linear system $|H_S + 2\delta|$ induces a morphism to \mathbb{P}^7 that realizes the blow-up map $S \rightarrow S_0$ by [42, Proposition 2.7] (in fact, we can get back S via a tangential projection from \mathbb{P}^7 at the point to which δ is contracted). The induced polarization of degree 12 on $S_0 \subseteq \mathbb{P}^7$ is denoted by H_{S_0} . Thus, $NS(S) \subseteq H^2(S, \mathbb{Z})$ is freely generated by the divisor classes $H_S + 2\delta$ and δ . Also note that H_T is a polarization of degree 12.

The algebraic lattice A_1 of $H^4(X, \mathbb{Z}) = H^4(\mathbb{P}^4, \mathbb{Z}) \oplus H^2(S, \mathbb{Z})$ is by definition the subgroup generated by algebraic cycles of co-dimension 2 in X , i.e., $A_1 = H^4(\mathbb{P}^4, \mathbb{Z}) \oplus NS(S)$. The orthogonal lattice of A_1 in $H^4(X, \mathbb{Z})$ with respect to the intersection form is evidently $T(S_0)$, the transcendental lattice of the K3 surface S_0 .

The intersection matrix of the lattice A_1 is

$$\begin{array}{c|ccc} & H^2 & H_S + 2\delta & \delta \\ \hline H^2 & 1 & 0 & 0 \\ H_S + 2\delta & 0 & 12 & 0 \\ \delta & 0 & 0 & -1 \end{array}$$

The algebraic lattice A_2 of $H^4(X, \mathbb{Z}) = H^4(W, \mathbb{Z}) \oplus H^2(T, \mathbb{Z})$ is $A_2 = H^4(W, \mathbb{Z}) \oplus NS(T)$, whereas the orthogonal lattice of A_2 in $H^4(X, \mathbb{Z})$ is $T(T)$, the transcendental lattice of the K3 surface T .

The intersection matrix of the lattice A_2 is

$$\begin{array}{c|ccc} & \Xi & \Pi & H_T \\ \hline \Xi & 2 & -1 & 0 \\ \Pi & -1 & 1 & 0 \\ H_T & 0 & 0 & 12 \end{array}$$

The isometry of Hodge structures decomposes as $\varphi_A \oplus \varphi_T : A_1 \oplus T(T_0) \rightarrow A_2 \oplus T(S_0)$. It induces the following commutative diagram by Corollary 1.1.5 :

$$\begin{array}{ccc} dA_1 & \xrightarrow{\varphi_A} & dA_2 \\ \cong \downarrow & & \downarrow \cong \\ dT(S_0)(-1) & \xrightarrow{\varphi_T} & dT(T)(-1) \end{array}$$

It is clear that dA_1 is isomorphic to \mathbb{Z}_{12} , with generator $\frac{H_S+2\delta}{12} \bmod A_1$. Moreover, if $b = |H_S + 2\delta| : S \rightarrow S_0$ is the blow-up map, then $dA_1 = d(H^4(\mathbb{P}^4, \mathbb{Z}) \oplus NS(S)) = d(\mathbb{Z}H^2) \oplus d(b^*NS(S_0)) \oplus d(\mathbb{Z}\delta) = d(b^*NS(S_0))$. Hence, $b^* : dNS(S_0) \rightarrow d(b^*NS(S_0)) = dA_1$ is an isometry given by $\frac{H_{S_0}}{12} \bmod NS(S_0) \mapsto \frac{H_S+2\delta}{12} \bmod A_1$. For A_2 , we have $dA_2 = d(H^4(W, \mathbb{Z})) \oplus dNS(T) = dNS(T)$, since $H^4(W, \mathbb{Z})$ is unimodular. In particular, its generator is $\frac{H_T}{12} \bmod A_2$.

Geometric action on cycles.

Next, we want to understand how cycles coming from σ and cycles coming from τ relate to each other in $H^4(X, \mathbb{Z})$. This is fundamental, for then we can find a lattice-theoretic property of the Hodge isometry that ultimately will help us to prove that S_0 is not isomorphic to T .

3.2.4 Proposition. *The canonical divisor of X can be written as $K_X = -5H + E = -3H' + F = -H - H'$ in $\text{Pic}(X)$. In particular, we have the quadro-quartic relations $H' = 4H - E$ and $H = 2H' - F$. Moreover, the following relations also hold in $CH^2(X)$:*

- (a) $HE = H_S$ and $H'E' = H_T$.
- (b) $4H^2 - H_S = 2H'^2 - H_T$.
- (c) $E^2 = -8H^2 + \delta + 5H_S$ and $E'^2 = -5E - 7\Pi + 3H_T$.

Proof. The canonical bundle of the flag manifold $Fl(1, 2; 5)$ is $-2H - 4H'$. Hence, $K_X = [K_{Fl(1,2;5)} \otimes \bigwedge^3 (\mathcal{O}(1, 1) \oplus \mathcal{O}(0, 1)^{\oplus 2})]_{|X} = -H - H'$. We also have $K_X = -5H + E = -3H' + E'$ by the blow-up formula for the canonical bundle. On the other hand, (a) is easily derived from [22, Example 8.3.9, iii)] and (b) follows from a straightforward computation using (a) and the equations $(-5H + E)(-H - H') = K_X^2 = (-3H' + E')(-H - H')$, $H^2 = 2HH' - E'H$ and $H'^2 = 4HH' - EH'$. We prove (c) now. We have $c_1(\mathbb{P}^4) = 5H$, $c_2(\mathbb{P}^4) = 10H^2$ and $c_1(S) = -\delta$. Hence, $c_2(X) - 10H^2 = \delta - E^2 = 8H^2 - 5H_S$ by [22, Example 15.4.3]. From the last equality we obtain the same expression for E^2 as in the statement. The computation of E'^2 is similar. \square

The morphism φ_A induced from the Hodge isometry of $H^4(X, \mathbb{Z})$ relates geometrically the algebraic lattices A_1 and A_2 as follows.

3.2.5 Lemma. $\varphi_A : dA_1 \rightarrow dA_2$ is given by multiplication by 7.

Proof. We systematically use the relations described in Proposition 3.2.4 above.

In $H^4(X, \mathbb{Z})$ we have

$$\begin{aligned} H'^2 &= (4H - E)^2 \\ &= 16H^2 - 8H_S + E^2 \\ &= 16H^2 - 8H_S + (-8H^2 + \delta + 5H_S) \\ &= 8H^2 - 3H_S + \delta \end{aligned}$$

Multiplying by 2 we get

$$\begin{aligned} 2H'^2 &= 16H^2 - 6H_S + 2\delta \\ &= 16H^2 - 7H_S + (H_S + 2\delta) \\ &= 28H^2 - 7H_S - 12H^2 + (H_S + 2\delta) \\ &= 7(4H^2 - H_S) - 12H^2 + (H_S + 2\delta) \\ &= 7(2H'^2 - H_T) - 12H^2 + (H_S + 2\delta) \\ &= 14H'^2 - 7H_T - 12H^2 + (H_S + 2\delta) \end{aligned}$$

From this we get $(H_S + 2\delta) - 12H^2 = 7H_T - 12H'^2$.

Therefore,

$$\begin{aligned} \varphi_A \left(\frac{H_S + 2\delta}{12} \bmod A_1 \right) &= \varphi_A \left(\frac{H_S + 2\delta}{12} - H^2 \bmod A_1 \right) \\ &= \varphi_A \left(\frac{(H_S + 2\delta) - 12H^2}{12} \bmod A_1 \right) \\ &= \frac{7H_T - 12H'^2}{12} \bmod A_2 \\ &= \frac{7H_T}{12} - H'^2 \bmod A_2 \\ &= \frac{7H_T}{12} \bmod A_2 \\ &= 7 \left(\frac{H_T}{12} \right) \bmod A_2. \end{aligned}$$

□

The main result of this section is

3.2.6 Theorem. *Let S_0 , or equivalently, T have Picard rank 1. Then $S_0 \not\cong T$.*

Proof. Assume that $\theta : S_0 \rightarrow T$ is an isomorphism. It naturally induces a Hodge isometry $H^2(T, \mathbb{Z}) \rightarrow H^2(S_0, \mathbb{Z})$, which can be decomposed as $\theta_A \oplus \theta_T : NS(T) \oplus T(T) \rightarrow NS(S_0) \oplus T(S_0)$. Moreover, since these surfaces have Picard rank 1, the degree 12 polarizations H_{S_0} and H_T are interchanged by θ_A .

The unimodularity of the K3 lattice induces the following diagram of discriminant groups by Corollary 1.1.5:

$$\begin{array}{ccc} dNS(T) & \xrightarrow{\theta_A} & dNS(S_0) \\ \cong \downarrow & & \downarrow \cong \\ dT(T)(-1) & \xrightarrow{\theta_T} & dT(S_0)(-1) \end{array}$$

Evidently, we can identify the isometry $\theta_T : dT(T)(-1) \rightarrow dT(S_0)(-1)$ with the isometry $dA_2 = dNS(T) \xrightarrow{\theta_A} dNS(S_0) \xrightarrow{b^*} dA_1$ given by $\frac{H_T}{12} \bmod A_2 \mapsto \frac{H_S+2\delta}{12} \bmod A_1$. We will continue to denote the latter by θ_A .

Observe that the composition $\theta_T \varphi_T$ gives a Hodge isometry of the transcendental lattice $T(S_0)$ of S_0 . However, Id and $-Id$ are the only Hodge isometries of transcendental lattices of algebraic K3 surfaces with odd Picard rank by [40, Lemma 4.1]. Consequently, $\theta_T \varphi_T = \pm Id$, and this equality still holds if we view it as an automorphism of $dT(S_0)$. But then the map $\theta_A \varphi_A \cong \theta_T \varphi_T$ is both $\pm Id$ and multiplication by 7 in $dA_1 \cong \mathbb{Z}_{12}$ by Lemma 3.2.5. Contradiction. \square

3.2.2 The family of Fano fourfolds K3-33

Construction, notation, and conventions.

3.2.7 Theorem. *Let $X = \mathcal{L}(G(2, 4) \times \mathbb{P}^5, \mathcal{U}_{G(2,4)}^*(0, 1) \oplus \mathcal{O}(1, 1) \oplus \mathcal{Q}_{G(2,4)}(0, 1))$. Then X is a smooth Fano fourfold of K3 type and the projections $\sigma : X \rightarrow G(2, 4)$ and $\tau : X \rightarrow \mathbb{P}^5$ are both realized as the blow-up of a four-dimensional smooth quadric in \mathbb{P}^5 along a surface of type $^{II}Z_E^{10}$. The corresponding minimal models are derived-equivalent K3 surfaces.*

Proof. See [10, Fano K3-33]. \square

Any member X of the family of fourfolds constructed in the previous theorem is called Fano fourfold K3-33.

The image of the second projection $\tau : X \rightarrow \mathbb{P}^5$ is a smooth quadric Q^4 . On the other hand, the first projection $\sigma : X \rightarrow G(2, 4)$ is surjective. Since the Grassmannian $G(2, 4)$ is embedded into \mathbb{P}^5 as a smooth quadric via the Plücker embedding, we will identify it with Q^4 as well. Let $S \subseteq Q^4$ (resp. $T \subseteq Q^4$) be the blow-up center of σ (resp. τ). We denote by d and d' (resp. δ and δ') the pair of skew (-1) -lines contained in S (resp. T). Also, write S_0 (resp. T_0) for the minimal model of S (resp. T).

We fix notation by means of the following diagram:

$$\begin{array}{ccccccc} & & E & \xleftarrow{j_1} & X & \xleftarrow{j_2} & E' \\ & \sigma_1 \swarrow & & & \sigma \swarrow & & \tau \swarrow \\ d \cup d' \hookrightarrow & S & \hookrightarrow & Q^4 & & Q^4 & \hookrightarrow T \hookrightarrow \delta \cup \delta' \\ & \swarrow & & & \searrow & & \\ & & & & & & \end{array}$$

The diagram above is a monoidal FM-diagram between S_0 and T_0 . We will show that it is non-trivial. More precisely, we will show that S_0 is not isomorphic to T_0 if their Picard rank is 1. This holds for almost all members of the family K3-33, according to the following proposition.

3.2.8 Proposition. *For a very general member X of the family Fano fourfolds K3-33 we have $\text{rk}(\text{Pic}(S_0)) = \text{rk}(\text{Pic}(T_0)) = 1$.*

Proof. We have $H^0(G(2, 4) \times \mathbb{P}^5, \mathcal{U}_{G(2,4)}^*(0, 1)) = V_4^* \otimes V_6^*$, $H^0(G(2, 4) \times \mathbb{P}^5, \mathcal{O}(1, 1)) = \wedge^2 V_4^* \otimes V_6^*$, and $H^0(G(2, 4) \times \mathbb{P}^5, \mathcal{Q}_{G(2,4)}(0, 1)) = V_4 \otimes V_6^*$. Hence, the family of Fano fourfolds K3-33 is parametrized by an open subscheme of $\mathbb{P} = \mathbb{P}((V_6^* \otimes V_4) \oplus (V_6^* \otimes \wedge^2 V_4^*) \oplus (V_6^* \otimes V_4^*))$. In fact, in [10, Fano K3-33] the surface S corresponding to a fourfold X belonging to the family is defined as the degeneracy locus of the morphism $\mathcal{E} \oplus \mathcal{E}' \oplus \mathcal{O}_{Q^4}(-1) = \mathcal{U}_{G(2,4)} \oplus$

$\mathcal{Q}_{G(2,4)}^* \oplus \mathcal{O}_{G(2,4)}(-1) \rightarrow V_6^* \otimes \mathcal{O}_{G(2,4)} = \mathcal{O}_{Q^4}^{\oplus 6}$ that is induced by the same element of \mathbb{P} defining X . Since the rational map $\mathbb{P} \dashrightarrow U_{Q^4}$ mapping a Fano fourfold X of type K3-33 to its corresponding surface S is dominant (see the proof of Proposition 2.3.7), the claim follows by Proposition 2.3.8. \square

In view of the previous proposition, we will assume from now on that S_0 , or equivalently, T_0 has Picard rank 1. In order to show that $S_0 \not\cong T_0$, we start by applying the blow-up formula for singular cohomology to X to get a composition of Hodge isometries:

$$H^4(Q^4, \mathbb{Z}) \oplus H^2(S, \mathbb{Z}) \xrightarrow{\sigma^* + j_{1*}\sigma_1^*} H^4(X, \mathbb{Z}) \xleftarrow{\tau^* + j_{2*}\tau_1^*} H^4(Q^4, \mathbb{Z}) \oplus H^2(T, \mathbb{Z})$$

We denote by H (resp. H') the pullback $\sigma^*(H)$ (resp. $\tau^*(H)$) on X . We use the same symbol to denote a codimension 2 cycle in X coming from S (resp. T) via $j_{1*}\sigma_1^*$ (resp. $j_{2*}\tau_1^*$). For example, $j_{1*}\sigma_1^*(d)$ is simply denoted by d . On the other hand, we have $H^4(Q^4, \mathbb{Z}) = CH^2(Q^4) = \mathbb{Z}P_1 \oplus \mathbb{Z}P_2$. We denote by P_1 and P_2 the pullbacks $\sigma^*(P_1)$ and $\sigma^*(P_2)$ on X . The notation $P'_1 = \tau^*(P_1)$ and $P'_2 = \tau^*(P_2)$ is not used, although it is left implicit. We thus have an identification $H^4(X, \mathbb{Z}) = H^4(Q^4, \mathbb{Z}) \oplus H^2(S, \mathbb{Z}) = H^4(Q^4, \mathbb{Z}) \oplus H^2(T, \mathbb{Z})$. Observe that this lattice is unimodular, although it is not even.

Denote by H_S (resp. H_T) the polarization of degree 10 of S (resp. T). Recall that the linear system $|H_S + d + d'|$ (resp. $|H_T + \delta + \delta'|$) induces a morphism to \mathbb{P}^7 that realizes the blow-up map $S \rightarrow S_0$ (resp. $T \rightarrow T_0$). The induced polarization of degree 12 on S_0 (resp. T_0) is denoted by H_{S_0} (resp. H_{T_0}). Thus, $NS(S) \subseteq H^2(S, \mathbb{Z})$ is freely generated by the divisor classes $H_S + d + d'$, d , and d' . We have a similar statement for T .

The algebraic lattice A_1 of $H^4(X, \mathbb{Z}) = H^4(Q^4, \mathbb{Z}) \oplus H^2(S, \mathbb{Z})$ is by definition the subgroup generated by algebraic cycles of codimension 2 in X , i.e., $A_1 = H^4(Q^4, \mathbb{Z}) \oplus NS(S)$. The orthogonal lattice of A_1 in $H^4(X, \mathbb{Z})$ with respect to the intersection form is evidently $T(S_0)$, the transcendental lattice of the K3 surface S_0 .

The intersection matrix of the lattice A_1 is

	P_1	P_2	$H_S + d + d'$	d	d'
P_1	1	0	0	0	0
P_2	0	1	0	0	0
$H_S + d + d'$	0	0	12	0	0
d	0	0	0	-1	0
d'	0	0	0	0	-1

We can define an algebraic lattice A_2 in $H^4(X, \mathbb{Z}) = H^4(Q^4, \mathbb{Z}) \oplus H^2(T, \mathbb{Z})$ in the same way as before, and for the induced isometry of Hodge structures $\varphi_A \oplus \varphi_T : A_1 \oplus T(S_0) \rightarrow A_2 \oplus T(T_0)$ we have the following commutative diagram of discriminant groups by Corollary 1.1.5:

$$\begin{array}{ccc}
 dA_1 & \xrightarrow{\varphi_A} & dA_2 \\
 \cong \downarrow & & \downarrow \cong \\
 dT(S_0)(-1) & \xrightarrow{\varphi_T} & dT(T_0)(-1)
 \end{array}$$

It is clear that dA_1 is isomorphic to \mathbb{Z}_{12} , with generator $\frac{H_S + d + d'}{12} \bmod A_1$. Moreover, if $b_S := |H_S + d + d'| : S \rightarrow S_0$ is the blow-up map, then $dA_1 = d(H^4(Q^4, \mathbb{Z}) \oplus NS(S)) = d(\mathbb{Z}P_1 \oplus \mathbb{Z}P_2) \oplus d(b_S^* NS(S_0)) \oplus d(\mathbb{Z}d \oplus \mathbb{Z}d') = d(b_S^* NS(S_0))$. Hence, $b_S^* : dNS(S_0) \rightarrow d(b_S^* NS(S_0)) = dA_1$ is an isometry given by $\frac{H_{S_0}}{12} \bmod NS(S_0) \mapsto \frac{H_S + d + d'}{12} \bmod A_1$. An analogous statement holds for A_2 .

3.2.3 Geometric action on cycles.

Next, we want to understand how cycles coming from σ and cycles coming from τ relate to each other in $H^4(X, \mathbb{Z})$. This is fundamental, for then we can find a lattice-theoretic property of the Hodge isometry that will ultimately help us to prove that S_0 is not isomorphic to T_0 .

3.2.9 Proposition. *The relations $K_X = -4H + E = -4H' + E' = -H - H'$ hold in $\text{Pic}(X)$. In particular, we have the cubo-cubic relations $H' = 3H - E$ and $H = 3H' - E'$. Moreover, the following relations also hold in $CH^2(X)$:*

- (a) $HE = H_S$ and $H'E' = H_T$.
- (b) $3H^2 - H_S = 3H'^2 - H_T$.
- (c) $E^2 = -5H^2 + d + d' + 4H_S$ and $E'^2 = -5H'^2 + \delta + \delta' + 4H_T$.

Proof. The canonical bundle of X is given by $K_X = [K_{G(2,4) \times \mathbb{P}^5} \otimes \bigwedge^5 (\mathcal{U}_{G(2,4)}^*(0,1) \oplus \mathcal{O}(1,1) \oplus \mathcal{Q}_{G(2,4)}(0,1))]_{|X} = -H - H'$. We also have $K_X = -4H + E = -4H' + E'$ by the blow-up formula for the canonical bundle. On the other hand, (a) is easily derived from [22, Example 8.3.9 iii)] and (b) follows from a straightforward computation using (a) and the equations $(-4H + E)(-H - H') = K_X^2 = (-4H' + E')(-H - H')$, $H^2 = 3HH' - E'H$ and $H'^2 = 3HH' - EH'$. We prove (c) now. We have $H^2 = P_1 + P_2$ in $CH^2(Q^4)$. Moreover, $S = 5H^2 = 5P_1 + 5P_2$ according to Theorem 2.3.1. Also, we have $c_1(Q^4) = 4H$, $c_2(Q^4) = 7H^2$ and $c_1(S) = -d - d'$. Hence, $c_2(X) = 7H^2 + d + d' - E^2 = 12H^2 - 4H_S$ by [22, Example 15.4.3]. From the last equality we obtain the same expression for E^2 as in the statement. By symmetry, we can derive an analogous expression for E'^2 . \square

The morphism φ_A induced from the Hodge isometry of $H^4(X, \mathbb{Z})$ relates geometrically the algebraic lattices A_1 and A_2 as follows.

3.2.10 Lemma. $\varphi_A : dA_1 \rightarrow dA_2$ is given by multiplication by 5.

Proof. We systematically use the relations described in Proposition 3.2.9 above.

In $H^4(X, \mathbb{Z})$ we have

$$\begin{aligned}
H'^2 &= (3H - E)^2 \\
&= 9H^2 - 6H_S + E^2 \\
&= 9H^2 - 6H_S + (-5H^2 + d + d' + 4H_S) \\
&= 4H^2 - 2H_S + d + d' \\
&= 4H^2 - 3H_S + (H_S + d + d') \\
&= 9H^2 - 3H_S - 5H^2 + (H_S + d + d') \\
&= 3(3H^2 - H_S) - 5H^2 + (H_S + d + d') \\
&= 3(3H'^2 - H_T) - 5H^2 + (H_S + d + d') \\
&= 9H'^2 - 3H_T - 5H^2 + (H_S + d + d')
\end{aligned}$$

From this we get

$$\begin{aligned}
 (H_S + d + d') &= -8H'^2 + 3H_T + 5H^2 \\
 &= -8H'^2 + 3H_T + 5(3H' - E')^2 \\
 &= -8H'^2 + 3H_T + 5(9H'^2 - 6H_T + E'^2) \\
 &= -8H'^2 + 3H_T + 5(9H'^2 - 6H_T + (-5H'^2 + \delta + \delta' + 4H_T)) \\
 &= -8H'^2 + 3H_T + 5(4H'^2 - 2H_T + \delta + \delta') \\
 &= -8H'^2 + 3H_T + 20H'^2 - 10H_T + 5(\delta + \delta') \\
 &= 12H'^2 - 7H_T + 5(\delta + \delta') \\
 &= 12H'^2 - 7H_T + 5(\delta + \delta') - 5H_T + 5H_T \\
 &= 12H'^2 - 12H_T + 5(H_T + \delta + \delta')
 \end{aligned}$$

Therefore,

$$\begin{aligned}
 \varphi_A \left(\frac{H_S + d + d'}{12} \bmod A_1 \right) &= \frac{12H'^2 - 12H_T + 5(H_T + \delta + \delta')}{12} \bmod A_2 \\
 &= H'^2 - H_T + 5 \left(\frac{H_T + \delta + \delta'}{12} \right) \bmod A_2 \\
 &= 5 \left(\frac{H_T + \delta + \delta'}{12} \right) \bmod A_2.
 \end{aligned}$$

This completes the proof of the lemma. \square

The main result of this section is

3.2.11 Theorem. *Assume S_0 , or equivalently, T_0 has Picard rank 1. Then $S_0 \not\cong T_0$.*

Proof. Let $\theta : S_0 \rightarrow T_0$ be an isomorphism. It induces a Hodge isometry $H^2(T_0, \mathbb{Z}) \rightarrow H^2(S_0, \mathbb{Z})$, which can be decomposed as $\theta_A \oplus \theta_T : NS(T_0) \oplus T(T_0) \rightarrow NS(S_0) \oplus T(S_0)$. Moreover, since these surfaces have Picard rank 1, the degree 12 polarizations H_{S_0} and H_{T_0} are interchanged by θ_A .

The unimodularity of the K3 lattice induces the following diagram of discriminant groups by Corollary 1.1.5:

$$\begin{array}{ccc}
 dNS(T_0) & \xrightarrow{\theta_A} & dNS(S_0) \\
 \cong \downarrow & & \downarrow \cong \\
 dT(T_0)(-1) & \xrightarrow{\theta_T} & dT(S_0)(-1)
 \end{array}$$

Evidently, we can identify the isometry $\theta_T : dT(T_0)(-1) \rightarrow dT(S_0)(-1)$ with the isometry $dA_2 \xleftarrow{b_T^*} dNS(T_0) \xrightarrow{\theta_A} dNS(S_0) \xrightarrow{b_S^*} dA_1$ given by $\frac{H_T + \delta + \delta'}{12} \bmod A_2 \mapsto \frac{H_S + d + d'}{12} \bmod A_1$. We will continue to denote the latter by θ_A . Now observe that the composition $\theta_T \varphi_T$ gives a Hodge isometry of the transcendental lattice $T(S_0)$ of S_0 . However, Id and $-Id$ are the only Hodge isometries of transcendental lattices of algebraic K3 surfaces with odd Picard rank by [40, Lemma 4.1]. Consequently, $\theta_T \varphi_T = \pm Id$, and this equality still holds if we view it as an

automorphism of $dT(S_0)$. But then the map $\theta_{A\varphi_A} \cong \theta_T\varphi_T$ is both $\pm Id$ and multiplication by 5 in $dA_1 \cong \frac{\mathbb{Z}}{12\mathbb{Z}}$, by Lemma 3.2.10. Contradiction. \square

3.3 Picard rank 2 cases

3.3.1 The family of Fano fourfolds K3-36

3.3.1 Theorem. *Let $X = \mathcal{L}(\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^4, \mathcal{O}(0, 0, 3) \oplus \mathcal{O}(1, 1, 1))$. Then the projection $\pi_{1,2} : X \rightarrow \mathbb{P}^1 \times \mathbb{P}^1$ is a fibration in Del Pezzo surfaces of degree 3 and the projection $\pi_{i,3} : X \rightarrow \mathbb{P}^1 \times \mathbb{P}^4$ realizes X as the blow-up of $\mathbb{P}^1 \times W_3$ along an elliptic K3 surface $S_{i,12}$ of degree 12 of Picard rank 2, where $i = 1, 2$, and W_3 is a smooth cubic threefold. The K3 surfaces $S_{1,12}$ and $S_{2,12}$ are the complete intersection of two divisors of type $(1, 1)$.*

Proof. See [10, Fano K3-36]. \square

3.3.2 Lemma. *If W is a Fano threefold of index 2 and Picard rank 1, and there is a correspondence $\mathbb{P}^1 \times W \leftarrow X \rightarrow \mathbb{P}^1 \times W$, where each morphism is a blow-up of $\mathbb{P}^1 \times W$ along a K3 surface, then the K3 surfaces are derived equivalent.*

Proof. The non-trivial singular cohomology groups of a Fano threefold W of index 2 and Picard rank 1 satisfy $H^0(W, \mathbb{Z}) = H^2(W, \mathbb{Z}) = H^4(W, \mathbb{Z}) = H^6(W, \mathbb{Z}) = \mathbb{Z}$. They are generated by the classes of W , a hyperplane section, a line, and a point, respectively. In particular, $H^{2,2}(\mathbb{P}^1 \times W) = H^4(\mathbb{P}^1 \times W, \mathbb{Z}) \cong H^4(W, \mathbb{Z}) \oplus H^2(W, \mathbb{Z})$ has no transcendental part. This implies that the K3 surfaces contained in the products $\mathbb{P}^1 \times W$ are derived-equivalent by [10, Lemma 3.4]. \square

3.3.3 Lemma. *Assume that the degree of W is $d = 3$ or 4 and assume that the K3 surfaces of Lemma 3.3.2 are the complete intersection of two divisors of type $(1, 1)$ and have Picard rank 2. Then the sublattices of the Néron-Severi groups of the K3 surfaces generated by the hyperplane class and the class of the fiber of the elliptic fibration induced by the projection $\mathbb{P}^1 \times W \rightarrow \mathbb{P}^1$ are of the form $\Lambda_{d,2d} = \begin{pmatrix} 4d & d \\ d & 0 \end{pmatrix}$.*

Proof. Let α be the pullback of the hyperplane class of \mathbb{P}^1 and let β be the pullback of the hyperplane class of W in the Chow ring $CH(\mathbb{P}^1 \times W)$. The degree of either K3 surface $S \subseteq \mathbb{P}^1 \times W$ is $(\alpha + \beta)^4 = 4d$. On the other hand, the class of an elliptic fiber of $S \rightarrow \mathbb{P}^1$ is given by $\alpha(\alpha + \beta)^2$, and its degree is $\alpha(\alpha + \beta)^3 = d$. Therefore, the sublattice $\langle H, F \rangle$ of $NS(S)$ generated by the hyperplane class and the class of the elliptic fiber is isometric to $\Lambda_{d,2d}$. \square

3.3.4 Proposition. *The K3 surfaces $S_{1,12}$ and $S_{2,12}$ are isomorphic.*

Proof. Let $S := S_{1,12} \subseteq \mathbb{P}^1 \times W_3$. Denote by H the hyperplane class of S and by F the class of the fiber of $S \rightarrow \mathbb{P}^1$. Then $\langle H, F \rangle = \Lambda_{3,6} = \Lambda_{3,0} \subseteq NS(S)$ by Lemma 3.3.3. In particular, $3 = n[NS(S) : \langle H, F \rangle]$, where n is the multisection index of S . Since 3 is prime, we have $n = 1$ or $n = 3$. If $n = 1$, then $FM(S) = \{S\}$ by Proposition 1.2.13. If $n = 3$, then $NS(S) = \Lambda_{3,0}$. In particular, $FM(S) = \{S\}$ by Theorem 1.2.9 and Corollary 1.2.12. The result follows by Lemma 3.3.2. \square

3.3.2 The family of Fano fourfolds K3-37

3.3.5 Theorem. *Let $X = \mathcal{L}(\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^5, \mathcal{O}(0, 0, 2)^{\oplus 2} \oplus \mathcal{O}(1, 1, 1))$. Then the projection $\pi_{1,2} : X \rightarrow \mathbb{P}^1 \times \mathbb{P}^1$ is a fibration in Del Pezzo surfaces of degree 4 and the projection $\pi_{i,3} : X \rightarrow \mathbb{P}^1 \times \mathbb{P}^4$ realizes X as the blow-up of $\mathbb{P}^1 \times W_4$ along an elliptic K3 surface $S_{i,16}$ of degree 16 of Picard rank 2, where $i = 1, 2$, and W_4 is the intersection of two smooth quadrics in \mathbb{P}^5 . The K3 surfaces $S_{1,16}$ and $S_{2,16}$ are the complete intersection of two divisors of type $(1, 1)$.*

Proof. See [10, Fano K3-37]. □

3.3.6 Proposition. *The K3 surfaces $S_{1,16}$ and $S_{2,16}$ are isomorphic.*

Proof. Let $S := S_{1,16} \subseteq \mathbb{P}^1 \times W_4$. Denote by H the hyperplane class of S and by F the class of the fiber of $S \rightarrow \mathbb{P}^1$. Then $\langle H, F \rangle = \Lambda_{4,8} = \Lambda_{4,0} \subseteq NS(S)$ by Lemma 3.3.3. In particular, $4 = n[NS(S) : \langle H, F \rangle]$, where n is the multisection index of S . Then we have $n = 1$, $n = 2$, or $n = 4$. If $n = 1$, then $FM(S) = \{S\}$ by Proposition 1.2.13. If $n = 2$ or $n = 4$, then $NS(S) = \Lambda_{n,k}$ for some integer k . In particular, $FM(S) = \{S\}$ by Theorem 1.2.9 and Corollary 1.2.12. The result follows by Lemma 3.3.2. □

3.3.3 The family of Fano fourfolds K3-39

3.3.7 Theorem. *Let $X = \mathcal{L}(\mathbb{P}^1 \times \mathbb{P}^2 \times G(2, 4), \mathcal{Q}_{\mathbb{P}^2}(0, 0, 1) \oplus \mathcal{O}(1, 1, 1))$. Then the projection $\pi_{1,2} : X \rightarrow \mathbb{P}^1 \times \mathbb{P}^2$ is a conic bundle, the projection $\pi_{1,3} : X \rightarrow \mathbb{P}^1 \times G(2, 4)$ realizes X as the blow-up of a conic in a hypersurface of bidegree $(1, 2)$, itself the blow-up of $G(2, 4)$ along a K3 surface S_8 of degree 8, and the projection $\pi_{2,3} : X \rightarrow \mathbb{P}^2 \times G(2, 4)$ realizes X as the blow-up of a K3 surface S_{22} of degree 22 and Picard rank 2 in $Y \subseteq \mathbb{P}^2 \times G(2, 4)$, where Y is the blow-up of $G(2, 4)$ along a conic c . The K3 surface S_{22} can be projected isomorphically into $G(2, 4) \subseteq \mathbb{P}^5$ as a K3 surface of degree 8 and Picard rank 2 containing the conic c . The K3 surfaces S_8 and S_{22} are derived equivalent.*

Proof. See [10, Fano K3-39]. □

3.3.8 Proposition. *The K3 surfaces S_8 and S_{22} are isomorphic.*

Proof. The K3 surface S_{22} is isomorphic to a K3 surface of degree 8 in \mathbb{P}^5 , has Picard rank 2, and contains the conic c . The result follows by Proposition 1.2.17, since S_8 and S_{22} are derived-equivalent. □

3.3.4 The family of Fano fourfolds K3-41

3.3.9 Theorem. *Let $X = \mathcal{L}(\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^3, \mathcal{O}(1, 1, 2))$. Then the projection $\pi_{1,2} : X \rightarrow \mathbb{P}^1 \times \mathbb{P}^1$ is a quadric fibration and the projection $\pi_{i,3} : X \rightarrow \mathbb{P}^1 \times \mathbb{P}^3$ realizes X as the blow-up of $\mathbb{P}^1 \times \mathbb{P}^3$ along an elliptic K3 surface $S_{i,12}$ of degree 12 of Picard rank 2, where $i = 1, 2$. The K3 surfaces $S_{1,12}$ and $S_{2,12}$ are the complete intersection of two divisors of type $(1, 2)$ and are derived-equivalent.*

Proof. See [10, Fano K3-41]. □

3.3.10 Proposition. *The K3 surfaces $S_{1,12}$ and $S_{2,12}$ are isomorphic.*

Proof. Let α be the pullback of the hyperplane class of \mathbb{P}^1 and let β be the pullback of the hyperplane class of \mathbb{P}^3 in the Chow ring $CH(\mathbb{P}^1 \times \mathbb{P}^3)$. The degree of $S := S_{1,12} \subseteq \mathbb{P}^1 \times \mathbb{P}^3$ is $(\alpha + 2\beta)^2(\alpha + \beta)^2 = 12$. On the other hand, the class of an elliptic fiber of $S \rightarrow \mathbb{P}^1$ is given by $\alpha(\alpha + 2\beta)^2$, and its degree is $\alpha(\alpha + 2\beta)^2(\alpha + \beta) = 4$. Therefore, the sublattice $\langle H, F \rangle$ of $NS(S)$ generated by the hyperplane class H and the class of the elliptic fiber F is isometric to $\Lambda_{4,6} = \Lambda_{4,2}$. In particular, $4 = n[NS(S) : \langle H, F \rangle]$, where n is the multisection index of S . Then we have $n = 1$, $n = 2$, or $n = 4$. If $n = 1$, then $FM(S) = \{S\}$ by Proposition 1.2.13. If $n = 2$ or $n = 4$, then $NS(S) = \Lambda_{n,k}$ for some integer k . In particular, $FM(S) = \{S\}$ by Theorem 1.2.9 and Corollary 1.2.12. The result follows since $S_{1,12}$ and $S_{2,12}$ are derived-equivalent. \square

3.3.5 The family of Fano fourfolds K3-46

3.3.11 Theorem. *Let $X = \mathcal{L}(\mathbb{P}^1 \times \mathbb{P}^3 \times \mathbb{P}^5, \mathcal{Q}_{\mathbb{P}^3}(0, 0, 1) \oplus \mathcal{O}(0, 1, 1) \oplus \mathcal{O}(1, 1, 1))$. Then the projection $\pi_{1,2} : X \rightarrow \mathbb{P}^1 \times \mathbb{P}^3$ realizes X as the blow-up of $\mathbb{P}^1 \times \mathbb{P}^3$ along the blow-up of a K3 surface S_{16} of degree 16 at two points, the projection $\pi_{1,3} : X \rightarrow \mathbb{P}^1 \times \mathbb{P}^5$ realizes X as the blow-up of the complete intersection of two divisors of bidegree $(0, 2)$ and $(1, 2)$, itself the blow-up of a quadric in \mathbb{P}^5 along a K3 surface S_8 of degree 8 containing a line, and $\pi_{2,3} : X \rightarrow \mathbb{P}^3 \times \mathbb{P}^5$ realizes X as the blow-up of a K3 surface S_{26} of degree 26 and Picard rank 2 in $Y \subseteq \mathbb{P}^3 \times \mathbb{P}^5$, where Y is the blow-up of a quadric in \mathbb{P}^5 along a line. The K3 surfaces S_8 , S_{16} , and S_{26} are derived-equivalent.*

Proof. See [10, Fano K3-46]. \square

3.3.12 Proposition. *The K3 surfaces S_8 , S_{16} , and S_{26} are isomorphic.*

Proof. The K3 surface S_8 has degree 8 in \mathbb{P}^5 , has Picard rank 2, and contains a line. The result follows from Proposition 1.2.16, since S_8 , S_{16} , and S_{26} are derived-equivalent. \square

3.3.6 The family of Fano fourfolds K3-47

3.3.13 Theorem. *Let $X = \mathcal{L}(\mathbb{P}^1 \times \mathbb{P}^3 \times \mathbb{P}^3, \mathcal{O}(0, 1, 1)^{\oplus 2} \oplus \mathcal{O}(1, 1, 1))$. Then the projections $\pi_{1,2}, \pi_{1,3} : X \rightarrow \mathbb{P}^1 \times \mathbb{P}^3$ realize X as the blow-up of $\mathbb{P}^1 \times \mathbb{P}^3$ along the blow-up of a K3 surface S_{20} of degree 20 and Picard rank 2 at four points and the projection $\pi_{2,3} : X \rightarrow \mathbb{P}^3 \times \mathbb{P}^3$ realizes X as the blow-up of a co-dimension two linear section of $\mathbb{P}^3 \times \mathbb{P}^3$ along a K3 surface that projects isomorphically to a linear determinantal quartic surface S_4 in each \mathbb{P}^3 . The K3 surfaces S_4 and S_{20} are derived-equivalent.*

Proof. See [10, Fano K3-47]. \square

3.3.14 Corollary. *The K3 surfaces S_4 and S_{20} are isomorphic.*

Proof. The K3 surface S_4 is a linear determinantal quartic surface of Picard rank 2 in \mathbb{P}^3 . The result follows from Proposition 1.2.14, since S_4 and S_{20} are derived equivalent. \square

3.3.7 The family of Fano fourfolds K3-48

3.3.15 Theorem. *Let $X = \mathcal{L}(\mathbb{P}^1 \times Fl(1, 2; 4), \mathcal{O}(0, (0, 1)) \oplus \mathcal{O}(1, (1, 1)))$. Then the projection $\pi_{1,2} : X \rightarrow \mathbb{P}^1 \times \mathbb{P}^3$ realizes X as the blow-up of $\mathbb{P}^1 \times \mathbb{P}^3$ along a K3 surface of degree 14 and Picard rank 2, the projection $\pi_{1,3} : X \rightarrow \mathbb{P}^1 \times G(2, 4)$ realizes X as the blow-up of $\mathbb{P}^1 \times Q^3$ along a K3 surface of degree 16 and Picard rank 2, and $\pi_{2,3} : X \rightarrow Fl(1, 2; 4)$ realizes X*

as the blow-up of a K3 surface of degree 24 in $\mathbb{P}_{Q^3}(\mathcal{U}_{|Q^3}) = \mathcal{L}(Fl(1, 2; 4), \mathcal{O}(0, 1))$, where \mathcal{U} is the tautological bundle of $Q^4 = G(2, 4)$. The K3 surfaces S_{14} , S_{16} and S_{24} are derived equivalent.

Proof. See [10, Fano K3-48]. □

3.3.16 Proposition. *The K3 surfaces S_{14} , S_{16} and S_{24} are isomorphic.*

Proof. Let α be the pullback of the hyperplane class of \mathbb{P}^1 and let β be the pullbacks of the hyperplane class of Q^3 in the Chow ring $CH(\mathbb{P}^1 \times Q^3)$. The degree of $S := S_{16} \subseteq \mathbb{P}^1 \times Q^3$ is $16 = S(\alpha + \beta)^2$. On the other hand, the class of an elliptic fiber of $S \rightarrow \mathbb{P}^1$ is given by αS , and its degree is $\alpha S(\alpha + \beta) = \alpha\beta S$. Since the projection of S in $Q^3 \subseteq \mathbb{P}^4$ is a K3 surface of degree 6, we have $16 = S(\alpha + \beta)^2 = 2\alpha\beta S + \beta^2 S = 2\alpha\beta S + 6$. This implies that the degree of the elliptic fiber of has degree 5. Therefore, the sublattice $\langle H, F \rangle$ of $NS(S)$ generated by the hyperplane class H and the class of the elliptic fiber F is isometric to $\Lambda_{5,8} = \Lambda_{5,3}$. In particular, $5 = n[NS(S) : \langle H, F \rangle]$, where n is the multisection index of S . Since 5 is prime, we have $n = 1$ or $n = 5$. If $n = 1$, then $FM(S) = \{S\}$ by Proposition 1.2.13. If $n = 5$, then $NS(S) = \Lambda_{5,3}$. In particular, $FM(S) = \{S\}$ by Theorem 1.2.9 and Corollary 1.2.12. The result follows since S_{14} , S_{16} and S_{24} are derived-equivalent. □

3.3.8 The family of Fano fourfolds K3-55

3.3.17 Theorem. *Let $X = \mathcal{L}(\mathbb{P}^1 \times \mathbb{P}^4 \times \mathbb{P}^5, \mathcal{Q}_{\mathbb{P}^4}(0, 0, 1) \oplus \mathcal{O}(0, 2, 0) \oplus \mathcal{O}(1, 1, 1))$. Then the projection $\pi_{1,2} : X \rightarrow \mathbb{P}^1 \times \mathbb{P}^4$ realizes X as the blow-up of $\mathbb{P}^1 \times Q^3$ along a K3 surface S_{14} of degree 14 and Picard rank 2, the projection $\pi_{1,3} : X \rightarrow \mathbb{P}^1 \times \mathbb{P}^5$ is a divisorial contraction to a complete intersection of divisors of bidegrees $(0, 2)$ and $(1, 2)$, singular along a line, and the projection $\pi_{2,3} : X \rightarrow \mathbb{P}^4 \times \mathbb{P}^5$ realizes X as the blow-up of a \mathbb{P}^1 -bundle over Q^3 along a K3 surface S_{30} of degree 30. The K3 surfaces S_{14} and S_{30} are derived-equivalent.*

Proof. See [10, Fano K3-55]. □

3.3.18 Proposition. *The K3 surfaces S_{14} and S_{30} are isomorphic.*

Proof. Let α be the pullback of the hyperplane class of \mathbb{P}^1 and let β be the pullbacks of the hyperplane class of Q^3 in the Chow ring $CH(\mathbb{P}^1 \times Q^3)$. The degree of $S := S_{14} \subseteq \mathbb{P}^1 \times Q^3$ is $14 = S(\alpha + \beta)^2$. On the other hand, the class of an elliptic fiber of $S \rightarrow \mathbb{P}^1$ is given by αS , and its degree is $\alpha S(\alpha + \beta) = \alpha\beta S$. Since the projection of S in $Q^3 \subseteq \mathbb{P}^4$ is a K3 surface of degree 6, we have $14 = S(\alpha + \beta)^2 = 2\alpha\beta S + \beta^2 S = 2\alpha\beta S + 6$. This implies that the degree of the elliptic fiber of has degree 4. Therefore, the sublattice $\langle H, F \rangle$ of $NS(S)$ generated by the hyperplane class H and the class of the elliptic fiber F is isometric to $\Lambda_{4,7} = \Lambda_{4,3}$. In particular, $4 = n[NS(S) : \langle H, F \rangle]$, where n is the multisection index of S . Hence, we have $n = 1, 2$, or 4 . If $n = 1$, then $FM(S) = \{S\}$ by Proposition 1.2.13. If $n = 2$ or $n = 4$, then $NS(S) = \Lambda_{n,k}$ for some integer k . In particular, $FM(S) = \{S\}$ by Theorem 1.2.9 and Corollary 1.2.12. The result follows since S_{14} and S_{30} are derived-equivalent. □

3.4 Monoidal FM-diagrams

The study of the different families of Fano fourfolds of K3 type in relation to the problem of the triviality of monoidal FM-diagrams from the past sections is summarized in the following

3.4.1 Theorem. *Let X be a Fano fourfold of K3 type belonging to the family K3-N in [10], where $N = 32, 33, 36, 37, 39, 41, 46, 47, 48,$ or 55 . If $N = 32$ or 33 and the Picard rank of the corresponding K3 surfaces is 1, then the monoidal FM-diagram induced by X is non-trivial. If $N \neq 32$ and 33 and the Picard rank of the corresponding K3 surfaces is 2, then the monoidal FM-diagram induced by X is trivial. \square*

4. The cubo-cubic transformation of the smooth quadric fourfold is very special

In this chapter we classify special birational transformations of the three and four-dimensional smooth quadrics, Q^3 and Q^4 . It turns out that there is only one such example in each dimension. In the case of Q^3 , it is given by the linear system of quadrics passing through a rational normal quartic curve. In the case of Q^4 , it is given by the linear system of cubics passing through a non-minimal K3 surface of degree 10 with 2 skew (-1) -lines.

4.1 Basic facts on special rational maps

Let $\varphi : Y \dashrightarrow Z$ be a rational map projective varieties, where we assume that Y is factorial, hence normal. Then we can describe φ by a linear system. Indeed, by factoriality there exists a unique invertible sheaf \mathcal{L} that extends the pullback $(\varphi|_{\text{dom}(\varphi)})^*(\mathcal{O}_Z(1))$ to all of Y . Clearly, we can also extend the sections $(\varphi|_{\text{dom}(\varphi)})^*(s)$, $s \in H^0(Z, \mathcal{O}_Z(1))$, to sections of \mathcal{L} . The corresponding map $\varphi^* : H^0(Z, \mathcal{O}_Z(1)) \rightarrow H^0(Y, \mathcal{L})$ is injective and its image defines a linear system in $|H^0(Y, \mathcal{L})|$ that induces φ up to a projectivity. The base locus scheme \mathfrak{B} of φ is defined as the scheme-theoretic intersection of any collection of divisors that make up a basis of this linear system. We say that φ is special if \mathfrak{B} is a smooth subvariety of Y .

Recall that the graph Γ_φ of the rational map φ is defined as the closure of the graph of the morphism $\varphi|_{\text{dom}(\varphi)}$ in $Y \times Z$. It has the universal property that characterizes the blow-up of Y along the closed subscheme \mathfrak{B} (or more precisely, along the ideal sheaf corresponding to \mathfrak{B}). Denote by $\sigma : \text{Bl}_{\mathfrak{B}}(Y) \rightarrow Y$ the blow-up morphism. Then there exists a unique morphism $\tau : \text{Bl}_{\mathfrak{B}}(Y) \rightarrow Z$ such that $(\Gamma_\varphi, \pi_Y, \pi_Z) \cong (\text{Bl}_{\mathfrak{B}}(Y), \sigma, \tau)$. In fact, τ is induced up to a projectivity by the complete linear system $|\sigma^*\mathcal{L} \otimes \mathcal{O}(-E)|$, where E is the exceptional divisor of σ . We even say that the rational map φ is resolved by the morphisms σ and τ via the diagram $Y \xleftarrow{\sigma} \text{Bl}_{\mathfrak{B}}(Y) \xrightarrow{\tau} Z$. Moreover, if φ is a birational map and Z is factorial, then τ is the blow-up of Z along the base locus scheme \mathfrak{B}' of the inverse map φ^{-1} , for the isomorphism $Y \times Z \cong Z \times Y$ restricts to the graphs Γ_φ and $\Gamma_{\varphi^{-1}}$. Furthermore, the subschemes \mathfrak{B} and \mathfrak{B}' have codimension at least 2 by [26, Ch. V, Lemma 5.1].

4.2 Numerical constraints

Let $k > 2$ and let $\varphi : Q^k \dashrightarrow Q^k$ be a special birational transformation whose base locus scheme is a smooth variety \mathfrak{B} . Set $X := \text{Bl}_{\mathfrak{B}}(Q^k)$. As before, we denote by σ the blow-up morphism $X \rightarrow Q^k$, by E its exceptional divisor, and by τ the morphism extending φ to

X . We will also keep denoting by H (resp. H') the pullback $\sigma^*(H)$ (resp. $\tau^*(H)$) on X . Observe that since \mathfrak{B} is smooth and $\text{Pic}(Q^k) = \mathbb{Z}H$, we can decompose the Picard group of X as $\text{Pic}(X) = \mathbb{Z}H \oplus \mathbb{Z}E$.

Let \mathfrak{B}' be the base locus scheme of φ^{-1} , which is not necessarily smooth, reduced, or irreducible. Then τ is the blow-up of Q^k along \mathfrak{B}' by our discussion above. Moreover, if $E' := \tau^{-1}(\mathfrak{B}')$ is the exceptional divisor of τ , then [19, Proposition 1.3] implies that E' is an irreducible divisor. Also, we have $E' = \mathfrak{b}(E'_{red})$ as effective Cartier divisors for some positive integer \mathfrak{b} .

According to our setting, relations $H' = nH - E$ and $H = mH' - E' = mH' - \mathfrak{b}E'_{red}$ hold in $\text{Pic}(X)$ for some positive integers n and m . Write $r := \dim(\mathfrak{B})$ and $r' := \dim(\mathfrak{B}')$.

4.2.1 Proposition. *We have $\mathfrak{b} = 1$. In other words, E' is reduced. Moreover, we have*

- (a) *If $\Omega = Q^k - \text{Sing}(\mathfrak{B}'_{red})$, then $\mathfrak{B}' \cap \Omega = \mathfrak{B}'_{red} \cap \Omega$, i.e., \mathfrak{B}' is generically reduced.*
- (b) *$\text{Pic}(X) = \mathbb{Z}H \oplus \mathbb{Z}E = \mathbb{Z}H' \oplus \mathbb{Z}E'$.*
- (c) *$K_X = -kH + (k - r - 1)E = -kH' + (k - r' - 1)E'$ in $\text{Pic}(X)$.*
- (d) *$E = (mn - 1)H' - nE'$ and $E' = (mn - 1)H - mE$ in $\text{Pic}(X)$.*

Proof. We have $\mathfrak{b}E'_{red} = mH' - H = (mn - 1)H - mE$. If $l \subseteq Q^k$ is a line and f is a fiber of $E \rightarrow \mathfrak{B}$, then $\mathfrak{b}((mn - 1)H - mE)l = mn - 1$ and $\mathfrak{b}((mn - 1)H - mE)f = m$. Thus, $\mathfrak{b} = 1$ follows. Finally, observe that the same kind of arguments used to prove [19, Proposition 2.1] also prove (a), (b), (c), and (d). \square

4.2.2 Corollary. *We have $n = \frac{r'+1}{k-r-1}$ and $m = \frac{r+1}{k-r'-1}$, or equivalently, $r = (k - r' - 1)m - 1$ and $r' = (k - r - 1)n - 1$.*

Proof. By Proposition 4.2.1 we have $-kH + (k - r - 1)E = -kH' + (k - r' - 1)E' = -k(nH - E) + (k - r' - 1)((mn - 1)H - mE)$, or equivalently, $-kH - (r + 1)E = -knH + (k - r' - 1)((mn - 1)H - mE) = (-kn + (k - r' - 1)(mn - 1))H - (k - r' - 1)mE = (-kn + (k - r' - 1)mn - (k - r' - 1))H - (k - r' - 1)mE$. In particular, $(k - r' - 1)m = r + 1$ and $-k = -kn + (k - r' - 1)mn - (k - r' - 1) = -kn + (r + 1)n - (k - r' - 1) = -(k - r - 1)n - (k - r' - 1)$, i.e., $r' = (k - r - 1)n - 1$. \square

4.2.3 Proposition. *If $k = 3$, then $(n, m) = (2, 2)$, i.e., φ is a quadro-quadric transformation of Q^3 , and $(r, r') = (1, 1)$; if $k = 4$, then $(n, m) = (3, 3)$, i.e., φ is a cubo-cubic transformation of Q^4 , and $(r, r') = (2, 2)$.*

Proof. This follows from Corollary 4.2.2 and the fact that \mathfrak{B} and \mathfrak{B}' have codimension at least 2 in Q^k . \square

In the following, \mathfrak{B} will be denoted by C if $k = 3$, and by S if $k = 4$. Let \mathfrak{d} be the degree of C and \mathfrak{g} its genus, and let \mathfrak{d} be the degree of S and $(\mathfrak{a}, \mathfrak{b})$ its bidegree.

4.2.4 Lemma. *We list the intersection numbers $H^i E^{k-i}$, where $k = 3, 4$.*

If $k = 3$, then

- (a) $H^3 = 2$.
- (b) $H^2 E = 0$.

$$(c) HE^2 = -d.$$

$$(d) E^3 = -2g + 2 - 3d.$$

If $k = 4$, then

$$(a) H^4 = 2.$$

$$(b) H^3E = 0.$$

$$(c) H^2E^2 = -d.$$

$$(d) HE^3 = -H_S K_S - 4d.$$

$$(e) E^4 = -9d - c_2(S) - 4H_S K_S.$$

Proof. See [43, Lemma 2.2.14 (ii) b)] for the case $k = 3$, and [45, Lemma 2.3 (ii)] for the case $k = 4$. \square

4.2.5 Proposition. *If $k = 3$, then C is a smooth rational quartic curve; if $k = 4$, then the degree d of the surface S is less than 18. Moreover, the following equations hold:*

$$(a) H_S K_S = 5d - 48.$$

$$(b) c_2(S) = 25d - 224.$$

$$(c) \pi = 3d - 23.$$

$$(d) 12\chi(\mathcal{O}_S) = K_S^2 + c_2(S) = d^2 + 23d - 2ab - 256.$$

Proof. Let k be arbitrary. We have $2 = H^k = (nH - E)^k$ since Q^k has degree 2. Now let us consider a general complete intersection of $k - 1$ hyperplane sections of Q^k . This is a smooth conic curve, and its strict transform with respect to τ is mapped onto a curve contained in the source Q^k via σ . The degree of this curve is $2m = (mH' - E')(H')^{k-1} = H(H')^{k-1} = H(nH - E)^{k-1}$. In the following, we will use Proposition 4.2.3 and Lemma 4.2.4. Assume $k = 3$. Then $2 = (2H - E)^3 = 14 - 3d + 2g$ and $4 = H(2H - E)^2 = 8 - d$. It follows that $d = 4$ and $g = 0$, and thus C is a rational quartic curve. Now assume $k = 4$. We consider a general complete intersection of two hyperplane sections of Q^4 . This is a smooth quadric surface in Q^4 , and its strict transform with respect to τ is mapped onto a surface contained in the source Q^4 via σ . The degree of this surface is $H^2H'^2 = H^2(3H - E)^2 = 2(3)^2 - d > 0$, and thus $d < 18$. Items (a) and (b) follow from the equations $2 = (3H - E)^4$ and $6 = H(3H - E)^3$. To prove (c), we use the adjunction formula $2\pi - 2 = H_S K_S + H_S^2$ together with (a). Finally, (d) follows from Noether's formula and Corollary 2.1.3. \square

We want to understand the geometry of the curve C and the surface S , and ultimately provide a model for them. The following proposition gives indeed a model for C and also basic qualitative information about the embedding of the surface S into Q^4 .

4.2.6 Proposition. *If $k = 3$, then $H^0(Q^3, \mathcal{I}_{C/Q^3}(2)) = \mathbb{C}^5$ and $H^0(Q^3, \mathcal{I}_{C/Q^3}(1)) = 0$. In particular, C is a rational normal quartic curve in Q^3 ; if $k = 4$, then $H^0(Q^4, \mathcal{I}_{S/Q^4}(3)) = \mathbb{C}^6$ and $H^0(Q^4, \mathcal{I}_{S/Q^4}(1)) = H^0(Q^4, \mathcal{I}_{S/Q^4}(2)) = 0$. In particular, S is not a complete intersection, it is non-degenerate, and it is not contained in a quadratic complex of Q^4 .*

Proof. We have $\mathbb{C}^{k+2} = \varphi^*(H^0(Q^k, \mathcal{O}_{Q^k}(1))) = H^0(Q^k, \mathcal{I}_{\mathfrak{B}/Q^k}(n)) = H^0(X, \mathcal{O}_X(nH - E))$ by [18, Theorem 7.1.3]. Now assume $j < n$. If $0 \neq s \in H^0(Q^k, \mathcal{I}_{\mathfrak{B}/Q^k}(j))$, then we can

define $k + 2$ independent non-zero sections $sx_0^{n-j}, sx_1^{n-j}, \dots, sx_{k+1}^{n-j} \in H^0(Q^k, \mathcal{I}_{\mathfrak{B}/Q^k}(n))$, where x_0, \dots, x_{k+1} are coordinates giving a basis of $H^0(Q^k, \mathcal{O}_{Q^k}(1))$. But then we see that the codimension 1 zero locus $V(s)$ must be contained in the base locus scheme \mathfrak{B} . This is absurd. In particular, if $k = 3$ or 4 , then we get the cohomology groups in the statement. Also notice that C is a rational normal curve since $C \subseteq \mathbb{P}^4$ has degree 4, it is isomorphic to \mathbb{P}^1 , and is non-degenerate. The statements about S are clear. \square

At this point it is possible to give a classification of special birational maps of the three-dimensional smooth quadric Q^3 .

4.2.7 Theorem. *Let $\varphi : Q^3 \dashrightarrow Q^3$ be a special birational transformation. Then φ is a quadro-quadric transformation, and its base locus scheme C is a rational normal quartic curve.*

Conversely, let $C \subseteq Q^3$ be any rational normal quartic curve. Then the linear system of quadrics in Q^3 passing through C induces a special birational transformation $\varphi : Q^3 \dashrightarrow Q^3$ with C as base locus scheme. Moreover, the base locus scheme of φ^{-1} is a rational normal quartic curve as well.

Proof. The first claim follows from Proposition 4.2.3 and Proposition 4.2.6. The converse statement is essentially proved in [36, Lemma 2.2]. We just remark that smoothness of the curve \mathfrak{B}'_{red} implies $\mathfrak{B}' = \mathfrak{B}'_{red}$ by (a) in Proposition 4.2.1. Thus, \mathfrak{B}' is also a rational normal quartic curve by symmetry. \square

The four-dimensional case needs more work. As a first step, we find the degree of the surface S using some results from the theory of congruences of lines in \mathbb{P}^3 .

4.2.8 Proposition. *S is a surface of degree $\mathfrak{d} = 10$.*

Proof. We have $3\mathfrak{d} - 23 = \pi > 0$ by (c) in Proposition 4.2.5. In particular, we have $8 \leq \mathfrak{d} \leq 17$. If $\mathfrak{d} = 8$ or 9 , we consider all pairs $(\mathfrak{a}, \mathfrak{b})$ of non-negative integers \mathfrak{a} and \mathfrak{b} such that $\mathfrak{a} + \mathfrak{b} = \mathfrak{d}$. For all such pairs we necessarily have $12|K_S^2 + c_2 = \mathfrak{d}^2 + 23\mathfrak{d} - 2\mathfrak{a}\mathfrak{b} - 256$ by (d) in Proposition 4.2.5. However, it can be easily checked that this divisibility condition never holds. On the other hand, if $\mathfrak{d} = 13, 14$ or 15 , we get $\pi > \pi_1(\mathfrak{d})$. This contradicts Theorem 2.2.1. Finally, if $\mathfrak{d} = 11, 12, 16$ or 17 , we get $\pi = \pi_1(\mathfrak{d})$. Then Theorem 2.2.2 implies that S is contained in a quadratic complex. This not possible by Proposition 4.2.6. \square

Using the classification of smooth surfaces of degree 10 in the four-dimensional smooth quadric, we can easily identify the geometric type of S . It turns out that S is a non-minimal K3 surface.

4.2.9 Proposition. *S is a surface of type ${}^{II}Z_E^{10}$. In particular, S contains two skew (-1) -lines whose contraction defines a K3 surface of genus 7.*

Proof. Recall that S is non-degenerate and it is not contained in a quadratic complex of Q^4 by Proposition 4.2.6. By substituting $\mathfrak{d} = 10$ in the formulas of Proposition 4.2.5, we get the following numerical data: $H_S K_S = 2$, $c_2(S) = 26$, and $\pi = 7$. These values can only belong to a surface of type Z_E^{10} by Theorem 2.3.1, which also has bidegree $(\mathfrak{a}, \mathfrak{b}) = (5, 5)$ and $K_S^2 = -2$ (see table 2.1, where the main invariants of non-degenerate surfaces of degree up to 10 in Q^4 are listed). Therefore, S is a surface of type ${}^{II}Z_E^{10}$. \square

The next proposition summarizes the characteristics of φ that we have found so far in the four-dimensional case.

4.2.10 Proposition. *Let $\varphi : Q^4 \dashrightarrow Q^4$ be a special birational transformation of Q^4 . Then φ is a cubo-cubic transformation, and its base locus scheme is a surface S of type ${}^{II}Z_E^{10}$.*

Proof. This is the content of Proposition 4.2.3 and Proposition 4.2.9. □

4.3 Effective classification

We are now interested to know if the converse of Proposition 4.2.10 holds, i.e., we ask the following question:

Do all surfaces of type ${}^{II}Z_E^{10}$ induce special birational transformations of the smooth quadric fourfold?

For the moment, we can prove a weaker statement.

4.3.1 Proposition. *Let $S \subseteq Q^4$ be a smooth surface of type ${}^{II}Z_E^{10}$. Then the linear system of cubic complexes passing through S induces a special birational map φ from Q^4 onto a quadric hypersurface Z in \mathbb{P}^5 of rank ≥ 3 whose base locus scheme is the surface S . In particular, $Bl_S(Q^4)$ is a Fano fourfold of index 1 anti-canonically embedded in $Q^4 \times Z \subseteq \mathbb{P}^5 \times \mathbb{P}^5$.*

Proof. The spinor bundles \mathcal{E} and \mathcal{E}' and the ideal sheaf \mathcal{I}_{S/Q^4} fit in the following short exact sequence by Proposition 2.3.2:

$$0 \rightarrow \mathcal{E}(-3) \oplus \mathcal{E}'(-3) \oplus \mathcal{O}_{Q^4}(-4) \rightarrow \mathcal{O}_{Q^4}(-3)^{\oplus 6} \rightarrow \mathcal{I}_{S/Q^4} \rightarrow 0.$$

An easy computation shows $H^0(Q^4, \mathcal{I}_{S/Q^4}(3)) = H^0(Q^4, \mathcal{O}_{Q^4}^{\oplus 6}) = \mathbb{C}^6$. Hence, the linear system $|H^0(Q^4, \mathcal{I}_{S/Q^4}(3))| \subseteq |H^0(Q^4, \mathcal{O}_{Q^4}(3))|$ defines a rational map $\varphi : Q^4 \dashrightarrow \mathbb{P}^5$. Moreover, since the map $\mathcal{O}_{Q^4}(-3)^{\oplus 6} \rightarrow \mathcal{I}_{S/Q^4}$ is surjective, the six cubic polynomial equations defining a basis of $H^0(Q^4, \mathcal{I}_{S/Q^4}(3))$ also generate a homogeneous ideal in the coordinate ring of Q^4 with associated ideal sheaf \mathcal{I}_{S/Q^4} , or equivalently, closed subscheme S . In other words, S is the scheme-theoretic intersection of the basis elements of the linear system defining φ . This shows that φ is special with base locus scheme S .

Let $X := Bl_S(Q^4)$ and E be the exceptional divisor of the blow-up $\sigma : X \rightarrow Q^4$. Then the linear system $|3H - E|$ defines a morphism $\tau : X \rightarrow \mathbb{P}^5$ extending φ via σ . If we substitute the numerical data of S into the equations of Lemma 4.2.4 (case $k = 4$), we get $(3H - E)^4 = 2$. Hence, φ is birational and the image $Z := \tau(X)$ is a four-dimensional non-degenerate variety of degree 2 in \mathbb{P}^5 , i.e., a quadric hypersurface of rank at least 3.

Finally, since the induced birational map $\varphi : Q^4 \dashrightarrow Z$ is special, the blow-up $Bl_S(Q^4)$ can be identified with the graph $\Gamma_\varphi \subseteq Q^4 \times Z$ of φ . In particular, $\mathcal{O}_{Q^4 \times Z}(1, 1)$ restricts to the ample divisor $H + H'$ on $Bl_S(Q^4)$. But the anti-canonical divisor of $Bl_S(Q^4)$ can be rewritten as $-K_{Bl_S(Q^4)} = 4H - E = H + (3H - E) = H + H'$, so it is ample. Moreover, using the decomposition $Pic(Bl_S(Q^4)) = \mathbb{Z}H \oplus \mathbb{Z}E$ we can easily check that $H + H'$ is a primitive divisor. Hence, the index is 1. □

A quadric of rank ≥ 3 in \mathbb{P}^5 is normal by [26, Ch. II, Proposition 8.23 (b)]. However, if we want to get a true converse of Proposition 4.2.10 we need to show that Z is actually smooth. As a first step we introduce a variety that will help us understand the geometry of the map $\varphi : Q^4 \dashrightarrow Z$.

4.3.2 Definition. Let \mathcal{V} be a non-degenerate subvariety of \mathbb{P}^k and ℓ a line not contained in \mathcal{V} . We say that ℓ is a trisecant of \mathcal{V} if the scheme-theoretic intersection $\ell \cap \mathcal{V}$ is zero-dimensional of length ≥ 3 . The trisecant variety of \mathcal{V} (in \mathbb{P}^k) is defined as $\text{Trisec}(\mathcal{V}) := \bigcup \{\ell \text{ is a line in } \mathbb{P}^k : \ell \subseteq \mathcal{V} \text{ or } \ell \text{ is a trisecant of } \mathcal{V}\}$.

Observe that $\text{Trisec}(S) \subseteq Q^4$ since $S \subseteq Q^4$ and any line intersecting a quadric hypersurface in more than two points (counted with multiplicity) is actually contained in it.

4.3.3 Proposition. *We have $\dim(\text{Trisec}(S)) \geq 3$ and $S \subseteq \text{Trisec}(S) \subseteq Q^4$. In particular, S has a trisecant line.*

Proof. In [6, Theorem 1] I. Bauer provides the classification of smooth, non-degenerate, connected complex surfaces Σ in \mathbb{P}^5 which are not scrolls and such that $\dim(\text{Trisec}(\Sigma)) \leq 2$. In fact, the last condition is also equivalent to $\text{Trisec}(\Sigma) \cap \Sigma \neq \Sigma$ by [6, Corollary 1]. There are just eight cases in her classification, none of which can be our surface S . Thus, the trisecant variety of S must have dimension at least 3 and contains the surface S . \square

4.3.4 Proposition. *All trisecants of S necessarily intersect S in a zero-dimensional scheme of length exactly 3.*

Proof. Let c_1, \dots, c_6 be cubic equations giving a basis of $H^0(Q^4, \mathcal{I}_{S/Q^4}(3))$ and let q be the equation of the quadric Q^4 . Also, denote by C_i the cubic in \mathbb{P}^5 determined by c_i . Then the homogeneous ideal $\langle q, c_1, c_2, \dots, c_6 \rangle$ in $\mathbb{C}[z_0, \dots, z_5]$ cuts the surface S in \mathbb{P}^5 scheme-theoretically, i.e., $S = \bigcap_{i=1}^6 C_i \cap Q^4$. Now let ℓ be a trisecant line of S . If J is the homogeneous ideal of ℓ in $\mathbb{C}[z_0, \dots, z_5]$, then $\frac{\mathbb{C}[z_0, \dots, z_5]}{J + \langle c_i \rangle} \rightarrow \frac{\mathbb{C}[z_0, \dots, z_5]}{J + \langle q, c_1, c_2, \dots, c_6 \rangle}$ is surjective. Hence, the induced map on local rings $\mathcal{O}_{p, \ell \cap C_i} \rightarrow \mathcal{O}_{p, \ell \cap S}$ is surjective for all $p \in \text{supp}(\ell \cap S)$ and all i . In particular, $\text{length}(\ell \cap S) \leq \text{length}(\ell \cap C_i)$ if $\ell \not\subseteq C_i$. Of course, there is at least one i such that $\ell \not\subseteq C_i$. Otherwise, we would have $\ell \subseteq \bigcap_{i=1}^6 C_i \cap Q^4 = S$, which is not possible by the definition of trisecant line. Thus, for such i we have $3 \leq \text{length}(\ell \cap S) \leq \text{length}(\ell \cap C_i) \leq 3$, i.e., $\text{length}(\ell \cap S) = 3$. \square

We study now the structure of the birational morphism τ with elementary techniques from the minimal model program. A reference for this subject is [16].

4.3.5 Proposition. *The class of the strict transform of any trisecant line of S induces an extremal ray R of the Mori cone $\overline{NE}(X)$. The ray R is not nef and H' is a good supporting divisor for R . In fact, τ is the contraction morphism induced by R .*

Proof. For Q^4 the numerical and rational equivalence relations on algebraic cycles coincide and we have

$$CH^*(Q^4) = \frac{\mathbb{Z}[H, P]}{\langle H^3 - 2HP, P^2 - H^2P \rangle}.$$

Here H is the hyperplane class and P is a plane from one of the two rulings of the quadric Q^4 . The grading is 1 for H , and 2 for P . Since HP can be represented by the class of a line l in Q^4 , we have $N_1(Q^4) = CH^3(Q^4) = \frac{\langle H^3, HP \rangle}{\langle H^3 - 2HP \rangle} = \mathbb{Z}l$.

Let f be a fiber in the exceptional divisor E of the blow-up $\sigma : X \rightarrow Q^4$. We know that σ is the contraction morphism induced by the extremal ray spanned by f in $\overline{NE}(X)$. Hence, if we denote the pullback of l under σ by l again, we have $N_1(X)_{\mathbb{R}} = \mathbb{R}l \oplus \mathbb{R}f$.

If ℓ is a trisecant line of S , then the numerical class of its strict transform $\tilde{\ell}$ under σ is $l - 3f$ by Proposition 4.3.4. We define R to be the ray $\mathbb{R}^+[\tilde{\ell}] \subseteq \overline{NE}(X)$. Observe that R is K_X -negative since X is Fano, or simply because $K_X(l - 3f) = (-4H + E)(l - 3f) = -1$. Also, the ray R is contracted by τ since $H'(l - 3f) = (3H - E)(l - 3f) = 0$. Moreover, if a curve in X has class $al + bf$, $a, b \in \mathbb{R}$, $a \geq 0$, and is contracted by τ , then $H'(al + bf) = (3H - E)(al + bf) = 3a + b = 0$, implying that its numerical class is $a(l - 3f) \in R$. In particular, R is an extremal ray and τ is the contraction morphism induced by R . The statements about R and H' are clear. \square

4.3.6 Corollary. *The morphism τ is a divisorial contraction induced by the extremal ray R and its exceptional locus $Exc(R)$ is the strict transform $\widetilde{Trisec(S)}$ of the trisecant variety $Trisec(S)$ of S under σ .*

Proof. Evidently $\widetilde{Trisec(S)}$ is contained in the exceptional locus $Exc(R)$ of the extremal ray R . In particular, $3 \leq \dim(\widetilde{Trisec(S)}) \leq \dim(\underline{Exc(R)}) \leq 3$ by Proposition 4.3.3 and by the fact that τ is birational. Hence, $Exc(R) = \widetilde{Trisec(S)}$ is a prime divisor of X . \square

We denote $T := \tau(Exc(R)) \subseteq Z$.

4.3.7 Proposition. *T is irreducible of dimension 2.*

Proof. The dualizing sheaf of Z is the line bundle $\mathcal{O}_Z(-4)$, so the canonical divisor K_Z is Cartier. Therefore, we can write $K_X = \tau^*K_Z + \lambda Exc(R)$ in $Pic(X)$, $\lambda \in \mathbb{Z}$, for τ is an isomorphism outside $Exc(R)$. In fact, we have $\lambda = 1$, since $K_X R = -1 < 0$, $Exc(R)R < 0$, and $\tau^*Pic(Z) = \ker(\times R : Pic(X) \rightarrow \mathbb{Z})$. Thus, $-4H + E = K_X = \tau^*K_Z + Exc(R) = -4H' + Exc(R)$, which gives $Exc(R) = 8H - 3E$ in $Pic(X)$. Now we compute $H'^2 Exc(R)^2 = (3H - E)^2(8H - 3E)^2 = -10 \neq 0$ using Lemma 4.2.4. This implies that T has dimension 2 by [9, Proposition 2.7]. Finally, T is irreducible since $Exc(R)$ is. \square

If the morphism $Exc(R) \rightarrow T$ is equidimensional (all fibers are of dimension 1), then Proposition 4.3.7 implies that both Z and T are smooth by the well-known result of T. Ando [2, Theorem 2.3]. Nevertheless, it may very well happen that fiber dimension is non-constant; examples of this phenomenon are abundant. In any case, we know that if there exists a fiber of dimension 2 then it is necessarily isolated by Proposition 4.3.7. In fact, it is certain that two-dimensional fibers exist over singular points of Z or T by [3, Theorem 4.1]. Of course, then the singularities of Z and T are isolated. This implies that Z can have a nodal singularity at worst.

Fortunately, a classification of isolated two-dimensional fibers in fourfold Fano-Mori contractions, i.e., proper surjective maps with connected fibers $f : X \rightarrow Z$ from a manifold X onto a normal variety Z such that $-K_X$ is f -ample, is provided by M. Andreatta and J. A. Wiśniewski in the following

4.3.8 Theorem. *Let $\varphi : X \rightarrow Z$ be a birational Fano-Mori contraction from a smooth variety X of dimension 4 onto a normal variety Z (possibly affine). Let $F = \varphi^{-1}(z)$ be a (geometric) fiber of φ such that $\dim(F) = 2$. Assume that all other fibers of φ have dimension < 2 and all components of the exceptional locus E of φ meet F (this may be achieved by shrinking Z to an affine neighbourhood of z and restricting φ to its inverse image, if necessary).*

If φ is not divisorial then $E = F \cong \mathbb{P}^2$ and its normal bundle is $N_{F/X} = \mathcal{O}_{\mathbb{P}^2}(-1) \oplus \mathcal{O}_{\mathbb{P}^2}(-1)$. In this situation the flip of φ exists.

If E is a divisor then Z as well as $S := \varphi(E)$ are smooth outside of z . Moreover, outside of F the map φ is a simple blow-down of the divisor E to the surface $S \subseteq Z$. The scheme theoretic fiber structure over F is trivial, that is the ideal \mathcal{I}_F of F is equal to the inverse image of the maximal ideal of z , that is $\mathcal{I}_F = \varphi^{-1}(m_z)\mathcal{O}_X$.

The fiber F and its conormal bundle $\frac{\mathcal{I}_F}{\mathcal{I}_F^2}$ as well as the singularity of Z and S at z can be described as follows

F	$N_{F/X}^*$	$Sing(Z)$	$Sing(S)$
\mathbb{P}^2	$T_{\mathbb{P}^2}(-1) \oplus \mathcal{O}_{\mathbb{P}^2}(1)/\mathcal{O}_{\mathbb{P}^2}$	cone over Q^3	smooth
\mathbb{P}^2	$\mathcal{O}_{\mathbb{P}^2}^{\oplus 4}/\mathcal{O}_{\mathbb{P}^2}(-1)^{\oplus 2}$	smooth	cone over twisted cubic
quadric	spinor bundle \mathcal{E}^* from Q^4	smooth	non-normal

The quadric fiber can be singular, even reducible, and in the subsequent table we present a refined description of its conormal bundle. The last entry in the table provides information about the ideal of a suitable surface S .

quadric type	conormal bundle	ideal of S in $\mathbb{C}[[x, y, z, t]]$
$\mathbb{P}^1 \times \mathbb{P}^1$	$\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(1, 0) \oplus \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(0, 1)$	(xz, xt, yz, yt)
quadric cone	$0 \rightarrow N_{F/X}^* \rightarrow \mathcal{I}_{line}$	generated by 5 cubics
$\mathbb{P}^2 \cup_{line} \mathbb{P}^2$	$T_{\mathbb{P}^2}(-1) \cup (\mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(1))$	generated by 6 quartics.

Proof. See [4, Theorem in p. 3]. □

Therefore, in order to show that Z is smooth it suffices to discard the existence of plane fibers in $Exc(R) \rightarrow T$. We give a lattice-theoretic condition on the Picard group of the K3 surface S_0 that prevents this situation. Observe, however, that Z could be smooth even if there exist plane fibers, although T would necessarily be singular then.

We will need a famous intersection-theoretic formula, which takes in account positive dimensional residual intersections of cycles in the Chow ring of a smooth variety.

4.3.9 Proposition. *Let Y be a scheme, and let $X_i \hookrightarrow Y$ be regularly embedded subschemes for $1 \leq i \leq r$. Also consider a n -dimensional variety V of Y . Set $m := n - \sum_i \text{codim}_Y(X_i)$ and let Z be a connected component of the scheme-theoretic intersection of the X_i and V .*

Define $(X_1 \cdot X_2 \cdot \dots \cdot X_r \cdot V)^Z \in CH_m(Z)$ to be the part of $X_1 \cdot X_2 \cdot \dots \cdot X_r \cdot V$ that is supported on Z . Then

$$(X_1 \cdot X_2 \cdot \dots \cdot X_r \cdot V)^Z = \left\{ \prod_{i=1}^r c(N_{X_i/Y|Z}) \cap s(Z, V) \right\}_m.$$

If Z is regularly embedded in V , then we have

$$(X_1 \cdot X_2 \cdot \dots \cdot X_r \cdot V)^Z = \left\{ \frac{\prod_{i=1}^r c(N_{X_i/Y|Z})}{c(N_{Z/V})} \right\}_m.$$

Proof. See [22, Proposition 9.1.1]. □

4.3.10 Remark. The notation $\{\alpha\}_k$ for a cycle α in a scheme means that we just consider the degree of its k -dimensional part.

The cycle class $(X_1 \cdot X_2 \cdot \dots \cdot X_r \cdot V)^Z$ is called the equivalence of Z in $X_1 \cdot X_2 \cdot \dots \cdot X_r \cdot V$.

4.3.11 Proposition. *If Z_1, Z_2, \dots, Z_k are the connected components of $\bigcap_{i=1}^r X_i \cap V$, then $X_1 \cdot X_2 \cdot \dots \cdot X_r \cdot V$ is equal to the sum of the equivalences $(X_1 \cdot X_2 \cdot \dots \cdot X_r \cdot V)^{Z_i}$. In particular,*

$$\deg(X_1 \cdot X_2 \cdot \dots \cdot X_r \cdot V) = \sum_{i=1}^k \deg((X_1 \cdot X_2 \cdot \dots \cdot X_r \cdot V)^{Z_i}).$$

Proof. See [22, Proposition 9.1.2]. □

Finally, let us prove

4.3.12 Proposition. *Let S_0 be the minimal model of S in \mathbb{P}^7 and let H_{S_0} be the hyperplane class of S_0 . If $\text{Exc}(R) \rightarrow T$ has a plane fiber, then there exists a smooth elliptic curve $\mathfrak{E} \subseteq S_0$ of degree 5 such that the sublattice $\langle H_{S_0}, \mathfrak{E} \rangle \subseteq \text{Pic}(S_0)$ is saturated and the points to which d and d' are contracted are contained in a single (possibly singular) fiber of the associated elliptic fibration of S_0 . More generally, if $\text{Exc}(R) \rightarrow T$ is not equidimensional, then $\text{rk}(\text{Pic}(S_0)) > 1$.*

4.3.13 Remark. In Theorem 2.4.3 and with the help of `Macaulay2`, we provided an explicit example of a K3 surface of degree 12 in \mathbb{P}^7 containing a smooth elliptic curve of degree 5 such that its Picard group is generated by the hyperplane class and the class of the elliptic curve. Moreover, we verified that a random projection from two of its points induces a surface of type $^{II}Z_E^{10}$ in \mathbb{P}^5 .

Proof. Assume that there exists a two-dimensional fiber $F = \tau^{-1}(z)$, $z \in T \subseteq Z$. Let F_1 be any irreducible component of F . Observe that the restricted morphism $\sigma|_{F_1} : F_1 \rightarrow \sigma(F_1)$ is finite, since $\{0\} = \mathbb{R}^+[l - 3f] \cap \mathbb{R}^+[f] \subseteq \overline{NE}(X)$. Hence, $F_1 \not\subseteq E$. Indeed, if on the contrary $F_1 \subseteq E$, then we would have a surjective morphism $\sigma|_{F_1} : F_1 \rightarrow S$ exhibiting S_0 as an unirational surface. This is not the case. A second argument runs as follows. If we assume $F_1 \subseteq E$, then the surjective morphism $\sigma|_{F_1} : F_1 \rightarrow S$ must be flat by *miracle flatness*, since F_1 is Cohen-Macaulay by [26, Proposition Ch. II 8.23 (a)] and S is smooth. But this implies that $(\sigma|_{F_1})^* : \text{Pic}(S) \rightarrow \text{Pic}(F_1)$ is injective by [22, Example 1.7.4], since $\text{Pic}(S)$ is torsion-free. Thus, $3 \leq \text{rk}(\text{Pic}(S)) \leq \text{rk}(\text{Pic}(F_1))$, which is not possible for F .

Therefore, the morphism $\sigma|_{F_1} : F_1 \rightarrow \sigma(F_1)$ is birational.

Case 1. $F \cong \mathbb{P}^2$.

Let us prove that $\sigma(F) \subseteq Q^4$ is a plane that intersects S in a cubic curve. Indeed, we compute $\mathcal{O}_X(4H - E)|_F = \mathcal{O}_X(-K_X)|_F = \mathcal{O}_F(-K_F) \otimes \det(N_{F/X}) = \mathcal{O}_{\mathbb{P}^2}(3) \otimes \mathcal{O}_{\mathbb{P}^2}(-2) = \mathcal{O}_{\mathbb{P}^2}(1)$ and $\mathcal{O}_X(3H - E)|_F = 0$, since F is contracted to a point under $\tau = |3H - E|$. Hence, we have $\mathcal{O}_X(H)|_F = \mathcal{O}_{\mathbb{P}^2}(1)$ and $\mathcal{O}_X(E)|_F = \mathcal{O}_{\mathbb{P}^2}(3)$. This implies that $\sigma(F)$ has degree 1, i.e., it is a plane. Thus, σ_F is an isomorphism. In particular, the intersection $\mathcal{C} := S \cap \sigma(F) \cong E \cap F$ is a cubic in the plane $\sigma(F)$.

Now we compute the degree of the image of the cubic \mathcal{C} under the blow-up map $b_S : S \rightarrow S_0 \subseteq \mathbb{P}^7$ induced by the complete linear system $|H_S + K_S|$ on S . First observe that the Segre class of \mathcal{C} in $\sigma(F)$ is $s(\mathcal{C}, \sigma(F)) = 1 - 3H_{\mathcal{C}}t$, where we denote by $H_{\mathcal{C}}$ the restriction of H to \mathcal{C} . On the other hand, the total Chern class of the normal bundle of S in Q^4 is $c(N_{S/Q^4}) = 1 + (K_S + 4H_S)t + 50t^2$ by Proposition 2.1.1. Let $(S \cdot \sigma(F))^{\mathcal{C}}$ be the equivalence of \mathcal{C} for the intersection $S \cdot \sigma(F)$. Since \mathcal{C} is connected, we have $S \cdot \sigma(F) = (S \cdot \sigma(F))^{\mathcal{C}} =$

$\left\{c(N_{S/Q^4}|_{\mathcal{C}}) \frown s(\mathcal{C}, \sigma(F))\right\}_0$ by Proposition 4.3.9 and Proposition 4.3.11. Clearly,

$$\begin{aligned} c(N_{S/Q^4}|_{\mathcal{C}}) \frown (1 - 3H_{\mathcal{C}}t) &= c(N_{S/Q^4}|_{\mathcal{C}}) \frown (1 - 3H_S t)|_{\mathcal{C}} \\ &= [c(N_{S/Q^4}) \frown (1 - 3H_S t)]|_{\mathcal{C}} \\ &= [(1 + (K_S + 4H_S)t + 50t^2)(1 - 3H_S t)]|_{\mathcal{C}} \\ &= 1 + (H_S + K_S)|_{\mathcal{C}}t. \end{aligned}$$

Therefore,

$$\begin{aligned} 5 &= (5P_1 + 5P_2)P_i = S \cdot \sigma(F) \\ &= \left\{c(N_{S/Q^4}|_{\mathcal{C}}) \frown s(\mathcal{C}, \sigma(F))\right\}_0 \\ &= (H_S + K_S)\mathcal{C}. \end{aligned}$$

This implies that the degree of $b_S(\mathcal{C}) \subseteq S_0$ is 5. In particular, $K_S\mathcal{C} = 2$, and thus $\mathcal{C}^2 = -2$ by the adjunction formula, for plane cubics have arithmetic genus equal to 1.

Write $\mathcal{C} = b_S^*(\mathcal{D}) + xd + yd'$ in $\text{Pic}(S)$, where $\mathcal{D} \in \text{Pic}(S_0)$ and $x, y \in \mathbb{Z}$. Thus, $\mathcal{D} = b_{S^*}(\mathcal{C}) = b_S(\mathcal{C})$ in $\text{Pic}(S_0)$. Then $2 = K_S\mathcal{C} = (d + d')(b_S^*(\mathcal{D}) + xd + yd') = -x - y$ and $-2 = \mathcal{C}^2 = (b_S^*(\mathcal{D}) + xd + yd')^2 = \mathcal{D}^2 - x^2 - y^2$. We want to find the possible pairs (x, y) and compute \mathcal{D}^2 . To begin with, notice that d and d' cannot be components of the plane cubic \mathcal{C} simultaneously, for they are disjoint. If neither d nor d' are components of \mathcal{C} , then $d\mathcal{C} = -x$ and $d'\mathcal{C} = -y$ are non-negative, implying $(x, y) = (-1, -1)$, $(-2, 0)$, or $(0, -2)$. However, if one of these lines is a component of \mathcal{C} , for example d , then d' is not contained in the plane $\sigma(F)$. Clearly, $d'\mathcal{C} = -y = 0, 1$, or 2 , which also gives $(x, y) = (-1, -1)$, $(-2, 0)$, or $(0, -2)$. We thus have $\mathcal{D}^2 = 0$ or 2 . Assume $\mathcal{D}^2 = 2$. Since $H_{S_0}\mathcal{D} = 5$, we have $(H_{S_0} - 2\mathcal{D})^2 = 0$ and $H_{S_0}(H_{S_0} - 2\mathcal{D}) = 2$. Then the restriction of $\mathcal{O}_{S_0}(H_{S_0} - 2\mathcal{D})$ to each smooth curve in $|H_{S_0}|$ defines a g_2^1 . This is absurd, since the inclusion $S_0 \subseteq \mathbb{P}^7$ restricts to the canonical map on each smooth curve in $|H_{S_0}|$. Therefore, $\mathcal{D}^2 = 0$ and $d\mathcal{C} = d'\mathcal{C} = 1$.

We claim that \mathcal{D} is nef. To prove this, it suffices to verify that the class of any irreducible component of $b_S(\mathcal{C})$ has non-negative intersection with \mathcal{D} . This is certainly the case if $b_S(\mathcal{C})$ is a curve, for $\mathcal{D}^2 = 0$. Assume that \mathcal{C} is reducible, but it contains no (-1) -line. Then the decomposition of \mathcal{C} as an effective Cartier divisor in the plane $\sigma(F)$ is either $3l_1$, $2l_1 + l_2$, $l_1 + l_2 + l_3$, or $l_1 + \mathfrak{c}$, where l_1 , l_2 , and l_3 are three pairwise distinct lines and \mathfrak{c} is a smooth conic. If $\mathcal{C} = 3l_1$, then $-2 = \mathcal{C}^2 = 9l_1^2$, which is not possible. If $\mathcal{C} = 2l_1 + l_2$, then d and d' each intersect l_2 in a point away from l_1 , so $l_2^2 = -4$ by the adjunction formula. Hence, $-2 = \mathcal{C}^2 = (2l_1 + l_2)^2 = 4l_1^2$, which is also not possible. Therefore, $\mathcal{C} = l_1 + l_2 + l_3$ or $l_1 + \mathfrak{c}$. Observe that d and d' each intersect \mathcal{C} transversely at a point away from the intersection points of its components. If $\mathcal{C} = l_1 + l_2 + l_3$, then $\mathcal{D} = b_{S^*}(\mathcal{C}) = \mathfrak{r}_1 + \mathfrak{r}_2 + \mathfrak{r}_3$ in $\text{Pic}(S_0)$, where \mathfrak{r}_1 , \mathfrak{r}_2 , and \mathfrak{r}_3 are three smooth rational curves pair-wisely intersecting transversely at a point. This implies $\mathfrak{r}_i\mathcal{D} = 0$. If $\mathcal{C} = l_1 + \mathfrak{c}$, then $\mathcal{D} = b_{S^*}(\mathcal{C}) = \mathfrak{r}_1 + \mathfrak{r}_2$ in $\text{Pic}(S_0)$, where \mathfrak{r}_1 and \mathfrak{r}_2 are two smooth rational curves such that $\mathfrak{r}_1\mathfrak{r}_2 = 2$. This implies $\mathfrak{r}_i\mathcal{D} = 0$. Now assume that \mathcal{C} is reducible and contains a (-1) -line, for example d . Then the decomposition of \mathcal{C} as an effective Cartier divisor in the plane $\sigma(F)$ is either $3d$, $2d + l_1$, $d + 2l_1$, $d + l_1 + l_2$, or $d + \mathfrak{c}$, where l_1 and l_2 are distinct lines that are also different from d , and \mathfrak{c} is a smooth conic. By intersecting with d or d' we can discard the first three cases. Therefore, $\mathcal{C} = d + l_1 + l_2$ or $d + \mathfrak{c}$. Observe that d' intersects \mathcal{C} transversely at a point away from the intersection points of its components and away from d . If $\mathcal{C} = d + l_1 + l_2$, then $\mathcal{D} = b_{S^*}(\mathcal{C}) = \mathfrak{r}_1 + \mathfrak{r}_2$ in $\text{Pic}(S_0)$, where \mathfrak{r}_1 and \mathfrak{r}_2 are two smooth rational curves intersecting transversely at two points if d ,

\mathfrak{l}_1 , and \mathfrak{l}_2 are not concurrent, or they are tangent at a single point otherwise. Thus, $\mathfrak{r}_1\mathfrak{r}_2 = 2$. This implies $\mathfrak{r}_i\mathcal{D} = 0$. If $\mathcal{C} = d + \mathfrak{c}$, then $b_S(\mathcal{C})$ is a rational curve with an ordinary double point if d intersects transversely \mathfrak{c} , or it is a rational curve with a cusp if d is tangent to \mathfrak{c} , and thus there is nothing to do. The claim follows from the previous analysis by cases.

We now claim that \mathcal{D} is primitive in $\text{Pic}(S_0)$. Assume on the contrary that $\mathcal{D} = s\mathcal{D}'$ for some $\mathcal{D}' \in \text{Pic}(S_0)$ and $s > 0$. Then $sH_{S_0}\mathcal{D}' = 5$, so we have $s = 1$ or 5 . If $s = 5$, then we have $H_{S_0}\mathcal{D}' = 1$ and $\mathcal{D}'^2 = 0$. This implies that H_{S_0} is not base-point free, since H_{S_0} is ample and \mathcal{D}' is effective. This is absurd. Consequently $s = 1$ and the claim follows. Notice that \mathcal{D} being primitive in $\text{Pic}(S_0)$ is equivalent to $\langle \mathcal{D} \rangle$ being saturated in $\text{Pic}(S_0)$.

We conclude that \mathcal{D} is isotropic, nef and primitive. Hence, there exists a smooth elliptic curve $\mathfrak{E} \subseteq S_0$ such that $\mathfrak{E} = \mathcal{D}$ in $\text{Pic}(S_0)$ by [32, Ch. 2, Proposition 3.10]. In particular, $b_S(\mathcal{C})$ is a (possibly singular) fiber of the associated elliptic fibration of S_0 and d and d' are contracted to points in it.

Lastly, let us show that $\langle H_{S_0}, \mathfrak{E} \rangle$ is a saturated sublattice of $\text{Pic}(S_0)$. Indeed, let M be the saturation of $\langle H_{S_0}, \mathfrak{E} \rangle$ in $\text{Pic}(S_0)$. Then $5^2 = \text{disc}(\langle H_{S_0}, \mathfrak{E} \rangle) = |\text{disc}(M)|[M : \langle H_{S_0}, \mathfrak{E} \rangle]^2$. Hence, $[M : \langle H_{S_0}, \mathfrak{E} \rangle] = 1$ or 5 . Assume that the index is 5 . Then there exists $\zeta \in M$ such that $\zeta \notin \langle H_{S_0}, \mathfrak{E} \rangle$, but $5\zeta \in \langle H_{S_0}, \mathfrak{E} \rangle$. Write $5\zeta = \alpha H_{S_0} + \beta \mathfrak{E}$, where $\alpha, \beta \in \mathbb{Z}$. Then $25\zeta^2 = 12\alpha^2 + 10\alpha\beta$, so we have $5|\alpha$. Hence, $\eta := \zeta - \frac{\alpha}{5}H_{S_0} \in M$ satisfies $5\eta = \beta\mathfrak{E}$. Then $\eta \in \langle \mathfrak{E} \rangle$, implying $\zeta \in \langle H_{S_0}, \mathfrak{E} \rangle$. This is a contradiction. Therefore $[M : \langle H_{S_0}, \mathfrak{E} \rangle] = 1$ and the claim follows.

Case 2. $F \cong V_2 \subseteq \mathbb{P}^3$, where V_2 is a quadric of rank at least 2.

Let us prove that $\sigma(F) \subseteq Q^4$ is a quadric of the same type as F that intersects S in a sextic. Indeed, we compute $\mathcal{O}_X(4H - E)|_F = \mathcal{O}_{V_2}(1)$ using the adjunction formula and the identification $\det(\mathcal{E}^*) = \mathcal{O}_{Q^4}(1)$. As before, this implies $\mathcal{O}_X(H)|_F = \mathcal{O}_{V_2}(1)$ and $\mathcal{O}_X(E)|_F = \mathcal{O}_{V_2}(3)$.

Subcase 1. $F = F_1 \cup F_2$ is isomorphic to the union of two planes meeting along a line.

By restricting $\mathcal{O}_X(H)$ and $\mathcal{O}_X(E)$ to F_i , we have $\mathcal{O}_X(H)|_{F_i} = \mathcal{O}_{\mathbb{P}^2}(1)$ and $\mathcal{O}_X(E)|_{F_i} = \mathcal{O}_{\mathbb{P}^2}(3)$. In particular, $\sigma(F_1)$ and $\sigma(F_2)$ have degree 1, i.e., they are planes too. Hence, $\sigma|_{F_1}$ and $\sigma|_{F_2}$ are isomorphisms, and thus the intersections $\mathcal{C}_1 := S \cap \sigma(F_1) \cong E \cap F$ and $\mathcal{C}_2 := S \cap \sigma(F_2) \cong E \cap F$ are cubics. Moreover, $\sigma(F_1)$ and $\sigma(F_2)$ are distinct, since $\sigma : X - E \rightarrow Q^4 - S$ is an isomorphism and $F_1, F_2 \not\subseteq E$. Thus, they intersect in a proper linear subspace, which must be a line, since $F_1 \cap F_2 \cong \mathbb{P}^1$. In summary, $\sigma : F \rightarrow \sigma(F)$ is an isomorphism, and thus we can apply the argument of *Case 1* to show $\text{rkPic}(S_0) \geq 3 > 1$.

Subcase 2. F is isomorphic to a smooth quadric or a quadric cone.

The computation above implies that $\sigma(F)$ has degree 2, i.e., it is a quadric. Moreover, the finite birational map $\sigma : F \rightarrow \sigma(F)$ is an isomorphism, since $\sigma(F)$ is necessarily normal due to irreducibility. Hence, the intersection $\mathcal{C} := S \cap \sigma(F) \cong E \cap F$ is a sextic in $\sigma(F)$.

Let us compute the intersection number $K_S\mathcal{C}$. Since the sextic \mathcal{C} is a Cartier divisor in $\sigma(F)$ by construction, the Segre class of \mathcal{C} in $\sigma(F)$ is $s(\mathcal{C}, \sigma(F)) = 1 - 3H_{\mathcal{C}}t$. Moreover, \mathcal{C} is also

connected by [26, Ch. III, Corollary 7.9], so we have

$$\begin{aligned}
 10 &= (5P_1 + 5P_2)(P_1 + P_2) = S \cdot \sigma(F) \\
 &= (S \cdot \sigma(F))^{\mathcal{C}} \\
 &= \left\{ c(N_{S/Q^4|_{\mathcal{C}}}) \frown s(\mathcal{C}, \sigma(F)) \right\}_0 \\
 &= (H_S + K_S)\mathcal{C}.
 \end{aligned}$$

This implies $K_S\mathcal{C} = 4$.

Assume $rk(\text{Pic}(S_0)) = 1$ and write $\mathcal{C} = uH_S + vd + wd'$ in $\text{Pic}(S)$, where $u, v, w \in \mathbb{Z}$. Then we have the following system of equations

$$\begin{cases} H_S\mathcal{C} = H_S(uH_S + vd + wd') = 10u + v + w = 6 \\ K_S\mathcal{C} = (d + d')(uH_S + vd + wd') = 2u - v - w = 4. \end{cases}$$

It is easy to see that no integer solutions exist for this system. This is a contradiction. \square

A quadric in \mathbb{P}^5 with a nodal singularity at worst is factorial by Grothendieck's parafactoriality theorem (see [25, Chap. XI, Corollaire 3.14]). Hence, the base locus scheme \mathfrak{B}' of the inverse $\varphi^{-1} : Z \dashrightarrow Q^4$ is defined and τ is the blow-up of Z along \mathfrak{B}' . Its exceptional divisor E' is reduced by the same proof of Proposition 4.2.1. By Zariski's main theorem, we then have $\mathfrak{B}'_{red} = \{z \in Z : \dim(\tau^{-1}(z)) \geq 1\}$ and all fibers are connected. Therefore, $E' = \text{Exc}(R)$ and $\mathfrak{B}'_{red} = T$.

We prove now the main result of this Thesis.

4.3.14 Theorem. *Let $\varphi : Q^4 \dashrightarrow Q^4$ be a special birational map. Then φ is a cubo-cubic transformation, and its base locus scheme is a surface S of type ${}^{II}Z_E^{10}$.*

Conversely, let $S \subseteq Q^4$ be a surface of type ${}^{II}Z_E^{10}$ and let S_0 be its minimal model. Then the linear system $|H_S + K_S|$ induces a morphism $S \rightarrow \mathbb{P}^7$ realizing the blow-up of S_0 at two points p and p' . If H_{S_0} is the hyperplane class of $S_0 \subseteq \mathbb{P}^7$ and there is no smooth elliptic curve $\mathfrak{E} \subseteq S_0$ of degree 5 such that $\langle H_{S_0}, \mathfrak{E} \rangle \subseteq \text{Pic}(S_0)$ is saturated and the points p and p' belong to a (possibly singular) fiber of the associated elliptic fibration of S_0 , then the linear system of cubics passing through S induces a special birational transformation $\varphi : Q^4 \dashrightarrow Q^4$ with S as base locus scheme. If moreover $rk\text{Pic}(S_0) = 1$, then the base locus scheme T of φ^{-1} is a smooth surface of type ${}^{II}Z_E^{10}$ as well. In fact, the K3 surfaces S_0 and T_0 are non-isomorphic Fourier-Mukai partners. More precisely, T_0 is the moduli space of stable sheaves of S_0 with Mukai vector $(2, H_{S_0}, 3)$.

Proof. The first claim is just Proposition 4.2.10. The converse claim follows from Proposition 4.3.1 and Proposition 4.3.12. Assume that $rk\text{Pic}(S_0) = 1$. Then T is smooth, and thus $\mathfrak{B}' = T$ by (a) in Proposition 4.2.1. Let T_0 be the minimal model of T . Clearly, the K3 surfaces S_0 and T_0 are derived-equivalent by [10, Lemma 3.4]. Moreover, if H_{S_0} is the induced polarization of degree 12 on S_0 , then the Fourier-Mukai partners of S_0 are isomorphic to either S_0 or the moduli space of stable sheaves over S_0 with Mukai vector $(2, H_{S_0}, 3) \in \tilde{H}(S_0, \mathbb{Z})$ by [40, Proposition 1.10]. However, we can show $S_0 \not\cong T_0$ in the same way we proved Theorem 3.2.11. \square

4.4 A note on special transformations in dimension 4

In this section we show that if a birational transformation $\varphi : Q^4 \dashrightarrow Q^4$ can be resolved by blowing up a smooth surface $S \subseteq Q^4$, i.e., there exist morphisms $Q^4 \xleftarrow{\sigma} Bl_S(Q^4) \xrightarrow{\tau} Q^4$ such that σ is the blow-up and $\varphi = \tau\sigma^{-1}$, then φ is in fact special and S is its base locus scheme.

Let $X = Bl_S(Q^4)$ and let E be the exceptional divisor of the blow-up $\sigma : X \rightarrow Q^4$ along S . We keep denoting by H a hyperplane section of Q^4 and its own pullback under σ . Also, we write d for the degree of S . Since $Pic(X) = \mathbb{Z}H \oplus \mathbb{Z}E$, the morphism τ is given by a linear subsystem of $|nH - mE|$, where n and m are positive integers.

4.4.1 Lemma. $(nH - mE)^4 = 2$ and $\dim|nH - mE| = \dim|\mathcal{I}_{S/Q^4}^m(n)| = 5$, where \mathcal{I}_{S/Q^4}^m is the ideal sheaf of functions vanishing along S with multiplicity m . In particular, τ is given by the complete linear system $|nH - mE|$.

Proof. Since τ is birational, we have $(nH - mE)^4 = 2$. Let $\bar{\tau} : X \rightarrow \mathbb{P}^N$ be the morphism associated to the complete linear system $|nH - mE|$. In particular, the image $\bar{\tau}(X) \subseteq \mathbb{P}^N$ is four-dimensional and its degree is 2. Since a non-degenerate variety V in \mathbb{P}^N satisfies $\deg(V) \geq 1 + \text{codim}(V)$, we must necessarily have $N = 5$. Finally, note that $\sigma_*(\mathcal{O}_X(nH - mE)) = \sigma_*(\sigma^*(\mathcal{O}_{Q^4}(n)) \otimes \sigma^{-1}(\mathcal{I}_{S/Q^4}^m)) = \mathcal{I}_{S/Q^4}^m(n)$. Thus, $\dim|nH - mE| = \dim|\mathcal{I}_{S/Q^4}^m(n)| = 5$. \square

Let C be a general complete intersection of three hyperplane sections of the quadric Q^4 . Then C is a smooth conic. The strict transform of C under τ is denoted by C' and the degree of the image curve $\sigma(C') \subseteq Q^4$ is denoted by Δ .

4.4.2 Proposition. *We have the following arithmetic constraints:*

(a)

$$\begin{aligned} 2 &= 2n^4 - 6dn^2m^2 + 4nm^3(H_S K_S + 4d) - m^4(9d + c_2(S) + 4H_S K_S); \\ 2\Delta &= 2n^3 - 3dnm^2 + m^3(H_S K_S + 4d). \end{aligned}$$

(b) $m^2 \mid 2(n^3 - \Delta)$.

(c) $m \mid 8(n - 1)$.

(d) $(n - 4m)\Delta = 1 - 4m$.

(e) $dm^2 < 2n^2$.

(f) *If φ is not an isomorphism, then $0 < m < n < 4m$ and $1 < d < 2(\frac{n}{m})^2 < 32$.*

Proof. (a) It is enough to develop the equations $2 = (nH - mE)^4$ and $2\Delta = H(nH - mE)^3$ using Corollary 4.2.4.

(b) This is clear from the second diophantine equation in item (a).

(c) Since τ is an isomorphism near C' and C , the adjunction formula gives $-2 = K_X C' + \deg(N_{C'/X})$ and $\deg(N_{C'/X}) = \deg(N_{C/Q^4}) = 6$. Hence, $-8 = K_X C' = (-4H + E)(nH - mE)^3$. From this equation we get $E(nH - mE)^3 = 8(\Delta - 1)$, which implies that $m \mid 8(\Delta - 1) \dots$ **(A)**. Also, by item (b) we have $4m^2 \mid 8(\Delta - n^3) \dots$ **(B)**. Using conditions **(A)** and **(B)**, and the equality $8(\Delta - n^3) = 8(\Delta - 1) + 8(1 - n^3)$, we deduce that $m \mid 8(1 - n^3)$. In particular, $mn \mid 8(n - n^4) \dots$ **(C)**. On the other hand, note that the first equation in item

(a) implies that $m^2 \mid 2(1 - n^4)$. Hence, $4m^2 \mid 8(1 - n^4) \dots$ **(D)**. Using conditions **(C)** and **(D)**, and the equality $8(n - n^4) = 8(n - 1) + 8(1 - n^4)$, we can finally show that $m \mid 8(n - 1)$.

(d) We have $2 = (nH - mE)(nH - mE)^3 = 2n\Delta - mE(nH - mE)^3 = 2n\Delta - 8m(\Delta - 1)$. After manipulating some terms we get $(n - 4m)\Delta = 1 - 4m$.

(e) Consider a general complete intersection of two hyperplane sections of Q^4 , and its strict transform under τ . Then the degree of the image of the strict transform under σ is $H^2(nH - mE)^2 = 2n^2 - dm^2$, which is positive.

(f) If $d = 1$, i.e., S is a plane, then $H_S K_S = -3$ and $c_2(S) = 3$. Plugging this values in the first equation of item (a) gives $1 = n^4 - 3n^2m^2 + 2nm^3 = n(n^3 - 3nm^2 + 2m^3)$. Thus, $n = 1$ and $m = 0$. But this implies that φ must be an isomorphism, which is excluded by hypothesis. Therefore, we must have $d \geq 2$.

Assume that $m = 0$. Then $n = 1$ by item (d), and thus φ is an isomorphism. This is not possible. Hence, $m \neq 0$. Consequently, $2 \leq d < 2(\frac{n}{m})^2$ by item (e). This implies that $m < n$. Now observe that if $n > 4m$, then Δ is negative by item (d) (the integer $1 - 4m$ is positive only if $m = 0$), and this cannot be. Hence, $n \leq 4m$. Moreover, if $n = 4m$, then $1 = 4m$ by item (d) once more, which is absurd. Therefore, $d < 2(\frac{4m}{m})^2 = 32$. \square

From now on, we will assume that φ is not an isomorphism. Define k to be the number $\frac{8(n-1)}{m}$. Note that k is an integer by item (c) in Proposition 4.4.2. Moreover, we have $0 \leq 8(\frac{m-1}{m}) < k < 32(\frac{n-1}{n}) < 32$ by item (f) in Proposition 4.4.2,

4.4.3 Proposition. *In the following table we display the possible values for k, n, m and Δ .*

k	n	m	Δ	Case
16	3	1	3	A
20	11	4	3	B
24	7	2	7	C
24	13	4	5	D
28	15	4	15	E
31	63	16	63	F

Proof. We shall analyze three cases: $m = 1, m = 2$, and $m \geq 3$. For this matter, we use items from Proposition 4.4.2.

Case $m = 1$. We have $1 < n < 4$, i.e., $n = 2$ or 3 . Plugging $n = 2$ in the formula from item (d) gives non-integer Δ . Hence $n = 3$, and thus $\Delta = 3$ and $k = 16$. This is *Case A* in the table.

Case $m = 2$. We have $2 < n < 8$. Plugging $n = 3, 4, 5$, or 6 in the formula from item (d) gives non-integer values of Δ , whereas $n = 7$ gives $\Delta = 7$ and $k = 24$. We cannot rule this possibility using item (b). This is *Case C* in the table.

Case $m \geq 3$. Here we will consider the subcases $\Delta = 1, \Delta = 2, \Delta = 3$, and $\Delta \geq 4$. It will be useful to use the following expression for Δ derived from item (d):

$$\Delta = 1 + \frac{kn - k}{(32 - k)n - 32}$$

Subcase $\Delta = 1$. This subcase is ruled out by item (d).

Subcase $\Delta = 2$. We have $4 \leq n = \frac{32-k}{32-2k}$ and $14 \leq k < 16$. However, no pair of integers (k, n) exists subject to these conditions. This subcase is thus ruled out.

Subcase $\Delta = 3$. We have $4 \leq n = \frac{64-k}{64-3k}$ and $18 \leq k < 22$. The only pairs of integer (k, n) subject to this conditions are $(20, 11)$, and $(21, 43)$. For the former solution we get that $m = 4$. For the latter solution we get that $m = 16$. We cannot rule out the former solution using item (b), but we can use it to rule out the latter solution. Thus, we are left with $k = 20$, $n = 11$, $m = 16$ and $\Delta = 3$. This is *Case B* in the table.

Subcase $\Delta \geq 4$. We have $kn - k \geq 3((32 - k)n - 32)$, and thus $k \geq \frac{96n-96}{4n-1} = \frac{24(4n-1)-72}{4n-1} = 24 - \frac{72}{4n-1} \geq 24 - \frac{72}{15} \geq 20$.

Now we will treat each of the cases $20 \leq k < 32$ separately. Recall that we look at solutions $(n, m, \Delta) \geq (4, 3, 4)$.

If $k = 20$, then $\Delta = 2 + \frac{8n+12}{12n-32}$. The second term is an integer only if $n = 11$, in which case $\Delta = 3 < 4$. Thus, we rule out this case.

If $k = 21$, then $\Delta = 2 + \frac{10n+11}{11n-32}$. The second term is an integer only if $n = 43$, in which case $\Delta = 3 < 4$. Thus, we rule out this case.

If $k = 22$, then $\Delta = 3 + \frac{2n+42}{10n-32}$. The second term is never an integer. Thus, we rule out this case.

If $k = 23$, then $\Delta = 3 + \frac{5n+41}{9n-32}$. The second term is never an integer. Thus, we rule out this case.

If $k = 24$, then $\Delta = 4 + \frac{72}{8n-32}$. The second term is an integer only if $n = 5, 7$ or 13 . For $n = 5$ we get non-integer m , and for $n = 7$ we get $m = 2 < 3$. Hence, $n = 13, m = 4$, and $\Delta = 5$. This last case cannot be ruled out using item (b). This is *Case D* in the table.

If $k = 25$, then $\Delta = 4 + \frac{4n+71}{7n-32}$. The second term is never an integer. Thus, we rule out this case.

If $k = 26$, then $\Delta = 5 + \frac{2n+102}{6n-32}$. The second term is an integer only if $n = 5$. However, this implies that Δ is negative. Thus, we rule out this case.

If $k = 27$, then $\Delta = 6 + \frac{2n+133}{5n-32}$. The second term is an integer only if $n = 7$ or 55 . If $n = 7$, then m is non-integer. If $n = 55$, then $m = 16$ and $\Delta = 7$. But this is not possible by item (b). Thus, we rule out this case.

If $k = 28$, then $\Delta = 8 + \frac{196}{4n-32}$. The second term is an integer only if $n = 7, 9, 15$ or 57 . If $n = 7$, then Δ is negative. If $n = 9$, then m is non-integer. If $n = 57$, then $m = 16$ and $\Delta = 9$, but this is not possible by item (d). We are left with $n = 15, m = 4$, and $\Delta = 15$. This case cannot be ruled out using item (b). This is *Case E* in the table.

If $k = 29$, then $\Delta = 10 + \frac{2n+259}{3n-32}$. The second term is an integer only if $n = 11$ or 291 . If $n = 11$, then m is non-integer. If $n = 291$, then $m = 80$ and $\Delta = 11$. This is not possible by item (b). Thus, we rule out this case.

If $k = 30$, then $\Delta = 16 + \frac{450}{2n-32}$. The second term is an integer only if $n = 7, 11, 13$ or 15 . But then Δ is negative in all cases. Thus, we rule out this case.

If $k = 31$, then $\Delta = 32 + \frac{961}{n-32}$. The second term is an integer only if $n = 31, 33, 63$ or 993 . If $n = 31$, then Δ is negative. If $n = 33$, then m is non-integer. If $n = 993$, then $m = 256$ and $\Delta = 33$. This is not possible by item (d). If $n = 63$, then $m = 16$ and $\Delta = 63$. This case cannot be ruled out using item (b). This is *Case F* in the table. \square

4.4.4 Proposition. *The cases B, C, D, E, and F in the table of Proposition 4.4.3 are not possible.*

Proof. Elementary arithmetic considerations will suffice to rule out cases B, D, and E. More precisely, for each of these cases we substitute the values of n , m , and Δ into the equations from item (a) in Proposition 4.4.2, and then we analyze the corresponding diophantine equations.

Case B $= (n, m, \Delta) = (11, 4, 3)$. We have

$$83d = 915 + 56H_S K_S - 8c_2(S);$$

$$272d = 2656 + 64H_S K_S$$

From the second equation we see that d is even (divide it by 32). However, this implies that 915 is even by the first equation, which is not possible.

Case D $= (n, m, \Delta) = (13, 4, 5)$. We have

$$163d = 1785 + 18432H_S K_S - 8c_2(S);$$

$$23d = 274 + 4H_S K_S$$

From the second equation we see that d is even. However, this implies that 1785 is even by the first equation, which is not possible.

Case E $= (n, m, \Delta) = (15, 4, 15)$. We have

$$9869d = 6720 + 64H_S K_S$$

From this equation we get that d is divisible by 64, which is not possible, since $d < 32$.

We deal now with cases C and F. For this last two cases, we will get a contradiction for each of the following possibilities: $h^0(Q^4, \mathcal{I}_{S/Q^4}(3)) = 0$, $h^0(Q^4, \mathcal{I}_{S/Q^4}(3)) = 1$, or $h^0(Q^4, \mathcal{I}_{S/Q^4}(3)) \geq 2$.

Case C $= (n, m, \Delta) = (7, 2, 7)$.

Assume $h^0(Q^4, \mathcal{I}_{S/Q^4}(3)) = 0$ and consider the surjective morphism

$$\bigoplus_{\substack{k_1+k_2=7 \\ k_1, k_2 \geq 0}} H^0(Q^4, \mathcal{I}_{S/Q^4}(k_1)) \otimes H^0(Q^4, \mathcal{I}_{S/Q^4}(k_2)) \rightarrow H^0(Q^4, \mathcal{I}_{S/Q^4}^2(7)) = \mathbb{C}^6.$$

The last equality comes from Lemma 4.4.1. We have $H^0(Q^4, \mathcal{I}_{S/Q^4}(k)) = 0$ for all $k \leq 3$ by the assumption. Since $k_1 + k_2 = 7$ implies that either k_1 or k_2 must be ≤ 3 , the morphism is in fact trivial. This is a contradiction to the surjectivity.

Assume $h^0(Q^4, \mathcal{I}_{S/Q^4}(3)) = 1$ and let $0 \neq A \in H^0(Q^4, \mathcal{I}_{S/Q^4}(3))$. Then $H^0(Q^4, \mathcal{I}_{S/Q^4}(k)) = 0$ for $k \leq 2$. Indeed, if there exists $0 \neq s \in H^0(Q^4, \mathcal{I}_{S/Q^4}(k))$ then we can define six independent non-zero sections $sx_0^{3-k}, sx_1^{3-k}, \dots, sx_5^{3-k} \in H^0(Q^4, \mathcal{I}_{S/Q^4}(3))$, which is absurd. We have the surjective morphism

$$\bigoplus_{\substack{k_1+k_2=6 \\ k_1, k_2 \geq 0}} H^0(Q^4, \mathcal{I}_{S/Q^4}(k_1)) \otimes H^0(Q^4, \mathcal{I}_{S/Q^4}(k_2)) \rightarrow H^0(Q^4, \mathcal{I}_{S/Q^4}^2(6)).$$

Since $k_1 + k_2 = 6$ implies that either k_1 or k_2 must be ≤ 2 , or $k_1 = k_2 = 3$, the vector space $H^0(Q^4, \mathcal{I}_{S/Q^4}^2(6))$ is necessarily generated by the section A^2 . But then $A^2x_0, A^2x_1, \dots, A^2x_5$ clearly define a basis of $H^0(Q^4, \mathcal{I}_{S/Q^4}^2(7))$. This implies that the strict transform of the zero locus $V(A)$ of the section A is the support of the base locus scheme of the linear system $|7H - 2E|$, which defines τ by Lemma 4.4.1. This is a contradiction to the fact that τ is everywhere defined.

Assume $h^0(Q^4, \mathcal{I}_{S/Q^4}(3)) \geq 2$ and let $A, B \in H^0(Q^4, \mathcal{I}_{S/Q^4}(3))$ be two independent sections. Then A^2 and AB are also independent in $H^0(Q^4, \mathcal{I}_{S/Q^4}^2(6))$. We claim now that there exists an integer $0 \leq i \leq 5$ such that A^2x_i is not a linear combination of the $ABx_j, j = 0, 1, \dots, 5$. Indeed, assume on the contrary that $A^2x_i = ABL_i$, where L_i is a linear form for each $i = 0, 1, \dots, 5$. We have $\frac{A}{B} = \frac{L_0}{x_0} = \frac{L_1}{x_1}$, and thus $L_0 = x_0 \frac{L_1}{x_1}$. In particular, x_1 divides L_1 . But this means that $\frac{A}{B}$ is a scalar, contrary to the choice of A and B . The claim is thus proved. But this also means that the seven sections $ABx_0, ABx_1, ABx_2, ABx_3, ABx_4, ABx_5$, and A^2x_i are independent in $H^0(Q^4, \mathcal{I}_{S/Q^4}^2(7)) = \mathbb{C}^6$ for some $0 \leq i \leq 5$. This is impossible.

Case $F = (n, m, \Delta) = (63, 16, 63)$.

Assume $h^0(Q^4, \mathcal{I}_{S/Q^4}(3)) = 0$ and consider the surjective morphism

$$\bigoplus_{\substack{k_1 + \dots + k_{16} = 63 \\ k_1, \dots, k_{16} \geq 0}} H^0(Q^4, \mathcal{I}_{S/Q^4}(k_1)) \otimes \dots \otimes H^0(Q^4, \mathcal{I}_{S/Q^4}(k_{16})) \twoheadrightarrow H^0(Q^4, \mathcal{I}_{S/Q^4}^{16}(63)) = \mathbb{C}^6.$$

The last equality comes from Lemma 4.4.1. We have $H^0(Q^4, \mathcal{I}_{S/Q^4}(3)) = 0$ for $k \leq 3$. Notice that if $k_1 + \dots + k_{16} = 63$, then at least one of the k_i 's is ≤ 3 . Hence, the source vector space of the morphism is trivial. This is a contradiction to the surjectivity.

Finally, assume $h^0(Q^4, \mathcal{I}_{S/Q^4}(3)) \geq 1$ and let $0 \neq A \in H^0(Q^4, \mathcal{I}_{S/Q^4}(3))$. Then $A^{16} \in H^0(Q^4, \mathcal{I}_{S/Q^4}^{16}(48))$ and multiplication by A^{16} defines an injective map

$$\times A^{16} : H^0(Q^4, \mathcal{O}_{Q^4}(15)) = \mathbb{C}^{\binom{20}{5}} \rightarrow H^0(Q^4, \mathcal{I}_{S/Q^4}^{16}(63)) = \mathbb{C}^6.$$

This is absurd. □

In view of Proposition 4.4.4, there can be only one possibility:

$$\text{Case } A = (n, m, \Delta) = (3, 1, 3).$$

4.4.5 Theorem. *If a birational transformation $\varphi : Q^4 \dashrightarrow Q^4$ is not an isomorphism and can be resolved by blowing up a smooth surface $S \subseteq Q^4$, then φ is special with base locus scheme S .*

Proof. Knowing that $(n, m, \Delta) = (3, 1, 3)$ we proceed to prove that S is a surface of type $^{II}Z_E^{10}$ as in subsection §4.2. Then we conclude using Proposition 4.3.1. □

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Titre : Transformations spéciales des quadriques

Mots clés : Transformations birationnelles spéciales, Variétés de Fano de dimension quatre de type K3, Surfaces K3, Variétés cubiques spéciales de dimension quatre

Résumé : Dans cette thèse, nous étudions les transformations birationnelles spéciales des quadriques lisses. Nous obtenons un résultat de classification en dimensions 3 et 4. Dans ces deux cas, nous démontrons qu'il n'existe qu'un seul exemple. Pour la dimension 3, il est défini par le système linéaire de quadriques passant par une courbe rationnelle normale quartique. Pour la dimension 4, il est défini par le système linéaire de cubiques passant par une surface K3 non minimale de degré 10 avec 2 (-1) -droites disjointes qui n'est contenue dans aucune autre quadrique. Le lieu de base de la transformation inverse est en général une surface lisse du même type. De plus, nous montrons que les surfaces K3 correspondantes sont des partenaires de Fourier-Mukai non isomorphes.

Ces surfaces sont également liées aux cubiques de dimension 4 spéciales. Plus précisément, nous montrons qu'une cubique générale dans le diviseur de Hassett des cubiques spéciales de discriminante 14 contient une telle surface. Il s'agit du premier exemple d'une famille de surfaces non rationnelles caractérisant les cubiques dans ce diviseur.

L'étude des transformations birationnelles spéciales des quadriques est motivée par un exemple décrit par M. Bernardara, E. Fatighenti, L. Manivel, et F. Tanturri, qui ont fourni une liste de 64 nouvelles familles de variétés de Fano de type K3. De nombreux exemples dans leur liste donnent des variétés qui admettent des contractions birationnelles multiples, réalisées comme des éclatements des variétés de Fano le long des surfaces K3 non minimales. La nature des constructions implique que les surfaces K3 ont des catégories dérivées équivalentes. Nous répondons partiellement à la question naturelle : Pour quelles familles les surfaces K3 correspondantes sont-elles isomorphes, et pour quelles familles ne le sont-elles pas ?

Title: Special transformations of quadrics

Key words: Special birational transformations, Fano fourfolds of K3 type, K3 surfaces, Special cubic fourfolds

Abstract: In this thesis we study special self-birational transformations of smooth quadrics. We obtain a classification result in dimensions 3 and 4. In these two cases, we prove that there is only one example. In the case of dimension 3, it is given by the linear system of quadrics passing through a rational normal quartic curve. In the case of dimension 4, it is given by the linear system of cubic complexes passing through a non-minimal K3 surface of degree 10 with 2 skew (-1) -lines that is not contained in any other quadric. The base locus scheme of the inverse map is in general a smooth surface of the same type. Moreover, we prove that the corresponding pair of K3 surfaces are non-isomorphic Fourier-Mukai partners.

These surfaces are also related to special cubic fourfolds. More precisely, we show that a general cubic in the Hassett divisor of special cubic fourfolds of discriminant 14 contains such a surface. This is the first example of a family of non-rational surfaces characterizing cubics in this divisor.

The study of special birational transformations of quadrics is motivated by an example described by M. Bernardara, E. Fatighenti, L. Manivel, et F. Tanturri, who provided a list of 64 new families of Fano fourfolds of K3 type. Many examples in their list give varieties that admit multiple birational contractions realized as blow-ups of Fano manifolds along non-minimal K3 surfaces. The nature of the constructions implies that the corresponding K3 surfaces have equivalent derived categories. We partially answer the natural question: for which families the corresponding K3 surfaces are isomorphic, and for which families they are not?